The Lace expansion for $|\varphi|^4$

Recent developments in Constructive Field Theory

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Abstract

Akira Sakai has shown that a lace expansion exists for critical Ising and scalar $g\varphi^4$ lattice models and used it to prove that

$$\langle \varphi_a \varphi_b \rangle \sim \frac{c(g)}{|a-b|^{d-2}}$$

for $d > 5$ provided $g > 0$ is small.

With Tyler Helmuth and Mark Holmes we find a different more general lace expansion which exists for n-component $|\varphi|^4$ and the continuous time lattice Edwards model ($n = 0$). Using it we extend the results of Sakai to $n = 0, 1, 2$ component models.

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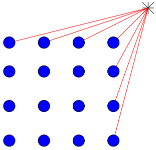

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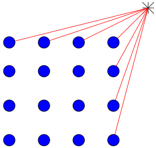

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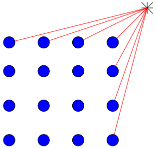

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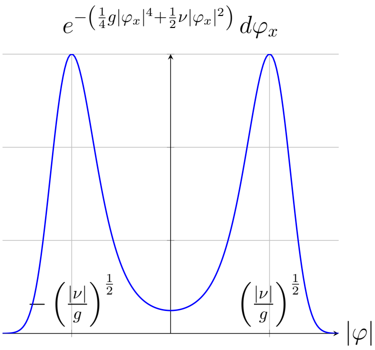

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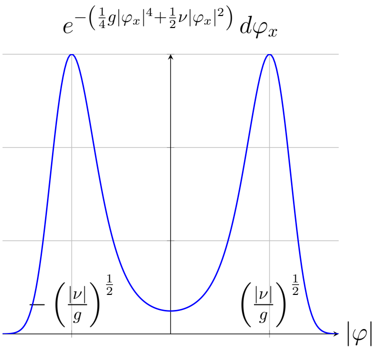

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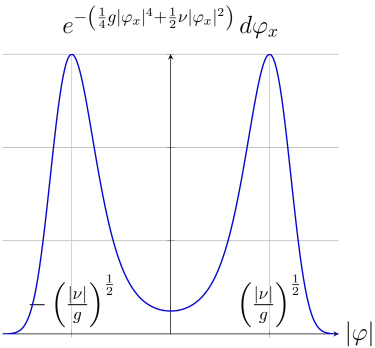

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2 point function

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$$\nu_c := \inf \left\{ \nu \;\middle|\; \langle \varphi_a \cdot \varphi_b \rangle^{\infty} \text{ summable in } b \right\}.$$

Theorem

Let $d > 4$ and $n \in \{0, 1, 2\}$. For $g > 0$ sufficiently small $\nu_c > -\infty$ and at $\nu = \nu_c$ there exists constant $C(g) > 0$ such that

$$\langle \varphi_a \cdot \varphi_b \rangle^\infty \sim \frac{C(g)}{|b-a|^{d-2}}, \qquad \text{as } |b-a| \to \infty.$$

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Rest of lecture: parts of the proof.

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local time $\tau = (\tau_x)_{x\in\Lambda}$ for X

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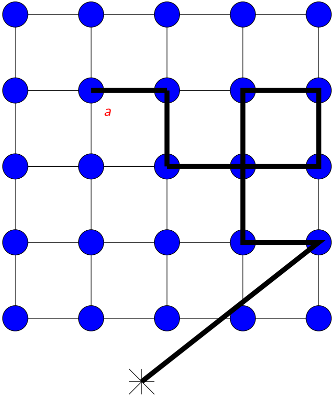

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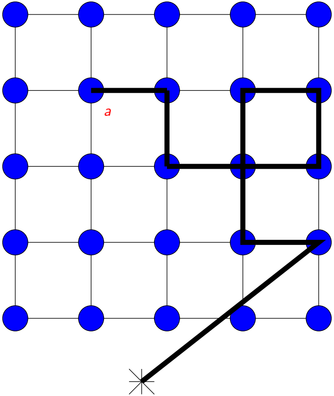

Random walk $s \mapsto \tilde{X}_s$ on $\mathbb{Z}^d$.

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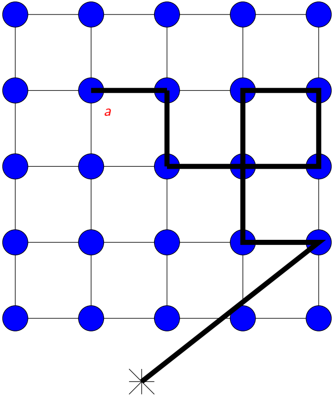
generator Δ: $(\varphi, \Delta\varphi) = -(\nabla\varphi, \nabla\varphi)$



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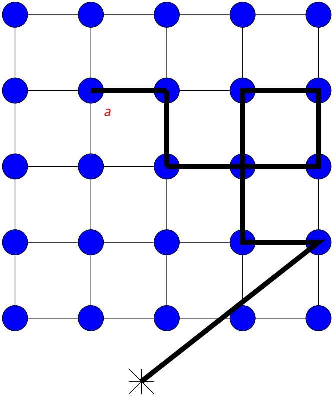

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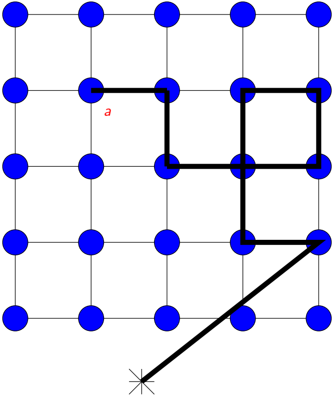

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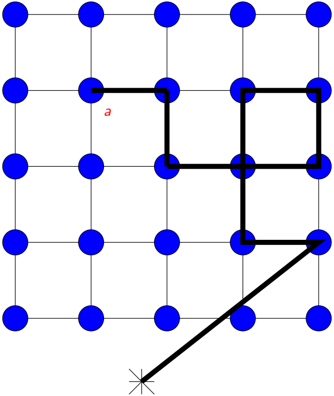

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Lace expansion: series representation for Π, terms bounded in terms of G.

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$\Rightarrow$

$$G \;\leq\; 2G^{\text{free}}, \qquad \nu \geq \nu_c.$$

Series for Π

$$z_{0,\ell} = z_{\ell,\ell} + \int_0^\ell ds\, L_s\, z_{s,\ell} + \int_0^\ell ds \int_s^\ell dt\, r_{s,t}\, z_{s,\ell}.$$

Series for Π

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Apply to

$$G(a,b) := \int d\ell\ \mathbb{E}_a \left[\frac{Z_{\tau_{[0,\ell]}}}{Z_0} \mathbb{1}_{\{X_\ell = b\}} \right].$$

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To get Π formula match terms with $G = G^{\text{free}} - G^{\text{free}} \Pi G$.

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- $z_{s,\ell} = \frac{z_{s,t} z_{t,\ell}}{z_{t,t}} + \left(\int_s^\ell ds' \int_t^\ell dt' \, r_{s',t'} \, \frac{z_{s,t'}}{z_{t,t'}} ds' dt' \right) z_{t,\ell}$ is the main new idea.

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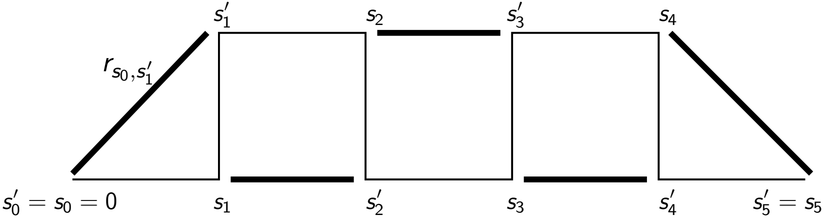

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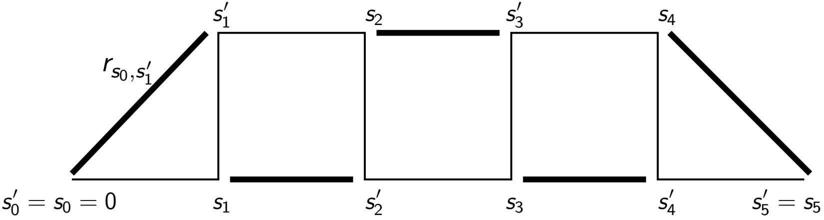
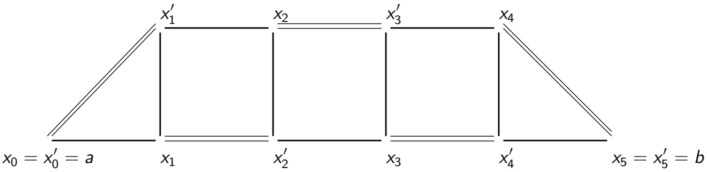


$$r_{s,t} := -\partial_s \partial_t \log Z_{\tau_{[s,t]}}.$$

Lace is bounded by G

$$|\Pi^{(i)}_{a,b}| \leq \sum_{x,x'} \prod_{\text{edges}} G_{\text{edge}}$$



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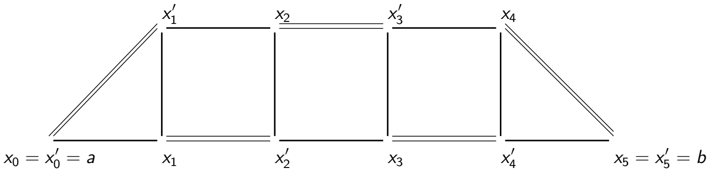

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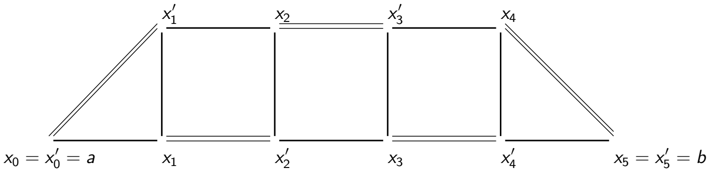

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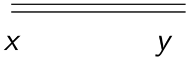
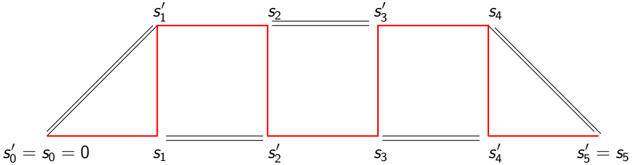


$= 16g^2 G^2(x,y) + 8g\,\mathbb{1}_{\{x=y\}}$

Detail



$$\int_{[s_4',\infty]} ds_5 \;\; \mathbb{E}\left[\frac{Z_{\tau_{[s_3,s_5']}}}{Z_{\tau_{[s_3,s_4']}}} \mathbb{1}_{X_{s_5}=b} \middle| \mathcal{F}_{s_4'} \right]$$

$$= \int_{[s_4',\infty]} ds_5 \;\; \mathbb{E}\left[\frac{Z_{\tau_{[s_3,s_4']}+\tau_{[s_4',s_5']}}}{Z_{\tau_{[s_3,s_4']}}} \mathbb{1}_{X_{s_5}=b} \middle| \mathcal{F}_{s_4'} \right]$$

$$\overset{\text{Markov}}{=} \; G_{\tau_{[s_3,s_4']}}(X_{s_4'}, b)$$

$$\overset{\text{Griffiths}}{\leq} \; G(X_{s_4'}, b).$$

The vertex function

For $s < t$ and $\tau \mapsto Z_\tau$, define the random variable

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$$= -2g \mathbb{1}_{\{X_s = X_t\}}, \qquad \text{forces a self-intersection.}$$

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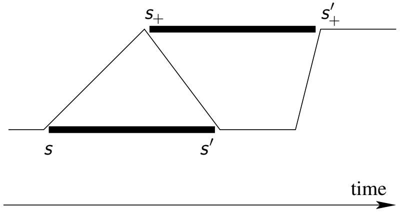
Formula that generates lace expansion



$$G_{\boldsymbol{\tau}_{[s,s']}}(X_{s'}, b) - G_{\mathbf{0}}(X_{s'}, b) =$$

$$\iint ds_+ \, ds'_+ \;\; \mathbb{E}\Big[r_{[s_+, s'_+]} \frac{Z_{\boldsymbol{\tau}_{[s,s'_+]}}}{Z_{\boldsymbol{\tau}_{[s,s']}}} G_{\boldsymbol{\tau}_{[s', s'_+]}}(X_{s'_+}, b) \Big| \mathcal{F}_{s'} \Big].$$

Lace expansion for Ising and φ^4

Sakai, A. (2007). Lace expansion for the Ising model.
Comm. Math. Phys., 272(2):283–344

Sakai, A. (2015). Application of the lace expansion to the φ^4 model.
Comm. Math. Phys., 336(2):619–648

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