Space-dependent renormalization group
and anomalous dimensions in a hierarchical
model for 3d CFT

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Partly joint work with Ajay Chandra (Imperial)
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Recent developments in Constructive Field Theory
Columbia University, March 13, 2018
Main references:


1 Introduction
2 The hierarchical continuum
3 The rigorous hierarchical space-dependent renormalization group
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(ACG2013) \(\rightarrow\) inhomogeneous RG for space-dependent couplings.

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e.g., \( g(x) = g + \delta g(x) \), with \( \delta g(x) \) a local perturbation such as test function.
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Used for generalizations of Zamolodchikov’s \( c \)-“Theorem”, study of scale vs. conformal invariance,\ldots
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Possibilities offered by the SDRG:

• Constructing correlation functions of scaling/continuum limits and corresponding probability measures, while simultaneously performing UV and large volume limits. Not just a stability bound.

• Constructing composite fields, e.g., the square $\Phi^2$ of the elementary field $\Phi$. Here $\Phi$ would be scaling limit of spin field and $\Phi^2$ that of the energy field.

• Showing Osterwalder-Schrader positivity with Fourier cutoffs, by emptying the interaction in a vanishing corridor around reflection hyperplane (A2015). QFT with defect/domain wall.

• Showing global/Möbius conformal invariance of scaling limit by controlling space-dependent UV cutoffs.

• Constructing explicit examples of holography or AdS/CFT correspondence.
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A touristic view of AdS/CFT:

\[ \hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{ \infty \} \cong \mathbb{S}^d. \]

The Möbius group \( \mathbb{M}(\mathbb{R}^d) \) is the group of bijective transformations of \( \hat{\mathbb{R}}^d \) generated by isometries, dilations and the unit sphere inversion \( J(x) = |x|^{-2}x \).

This is also the invariance group of the absolute cross-ratio
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CR(x_1, x_2, x_3, x_4) = \frac{|x_1 - x_3|}{|x_2 - x_4|}.
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Conformal ball model: \( \hat{\mathbb{R}}^d \cong \mathbb{S}^d \) seen as boundary of \( \mathbb{B}^{d+1} \) with metric
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ds = \frac{2}{|dx|} - \frac{1}{|x|^2}.
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Half-space model: \( \mathbb{R}^d \) seen as boundary of \( \mathbb{H}^{d+1} = \mathbb{R}^d \times (0, \infty) \) with metric
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Correspondence: \( f \in \mathbb{M}(\mathbb{R}^d) \leftrightarrow \) hyperbolic isometry of the interior \( \mathbb{B}^{d+1} \) or \( \mathbb{H}^{d+1} \), the Euclidean AdS space.
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A scalar field $O$ of scaling dimension $\Delta$ in a CFT on $\mathbb{R}^d$ has pointwise correlations which satisfy

$$\langle O(x_1) \cdots O(x_n) \rangle = (n! \prod_{i=1}^n |J_f(x_i)|^\Delta)^d \times \langle O(f(x_1)) \cdots O(f(x_n)) \rangle$$

for all $f \in \mathcal{M}(\mathbb{R}^d)$ and all collection of distinct points in $\mathbb{R}^d \setminus \{f^{-1}(\infty)\}$.

Here, $J_f(x)$ denotes the Jacobian of $f$ at $x$.

The AdS/CFT correspondence, discovered by Maldacena 1997 and made more precise by Gubser, Klebanov, Polyakov and Witten 1998, postulates a relation of the form:

$$\langle e^{\int_{\mathbb{R}^d j(x)O(x)dx} \rangle}_{\text{CFT}} = e^{-S[\phi_{\text{ext}}]}$$

where $S[\phi]$ is an action for a field $\phi(x, x_{d+1})$ on AdS space and $\phi_{\text{ext}}$ makes it extremal for a boundary condition $\phi(x, x_{d+1}) \sim |x_{d+1}|^{d-\Delta} j(x)$ when $x_{d+1} \to 0$. 
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AdS/CFT or holographic correspondence not yet known explicitly, i.e., exact $S[\phi]$ still mysterious. However, physicists have been experimenting with toy actions of the form:

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\int_{\mathbb{R}^{d+1}} \sqrt{\det g_{\mu\nu}} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 + \cdots \right\}
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where $m^2$ is related to $\Delta$ and is allowed to be (not too) negative. This gives an expansion for connected CFT correlations in terms of tree-level Feynman diagrams (Witten diagrams). The simplest "Mercedes logo" 3-point Witten diagram reproduces the correct CFT prediction $O(1)$ for $\langle O_1(x_1) O_2(x_2) O_3(x_3) \rangle$ by a calculation of Freedman, Mathur, Matusis and Rastelli 1999.
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Hierarchical models have a long history:

Dyson 1969, Wilson 1971 (the approximate recursion), . . .

But also Mandelbrot cascades (see 2013 IAS talk by Kupiainen), the branching Brownian motion used by Bramson and Zeitouni, Walsh-Fourier series in harmonic analysis, the setup used by Brydges, Evans and Imbrie which takes advantage of an additive group structure and Fourier analysis, . . .

the “God given” $p$-adic setup . . . where “God” is man called Alexander Ostrowski. Comes with a huge available knowledge base one can tap into . . . provided one has a gun to force number theorists to talk about $SO(d + 1, 1)$ instead of a general split reductive group over an arbitrary global field of characteristic zero.
The hierarchical continuum:

Let $p$ be an integer $>1$ (in fact a prime number). Let $L_k, k \in \mathbb{Z}$, be the set of cubes $\prod_{d=1}^{d} [a_i^p k, (a_i+1)^p k]$ with $a_1, ..., a_d \in \mathbb{N}_0$. The cubes of $L_k$ form a partition of the octant $[0, \infty)^d$. Hence $T = \bigcup_{k \in \mathbb{Z}} L_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $L_k$: 
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Hence $\mathcal{T} = \bigcup_{k \in \mathbb{Z}} \mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$: 
Picture for $d = 1$, $p = 2$
Forget $[0, \infty)^d$ and $\mathbb{R}^d$ and just keep the tree.
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A path representing an element $x \in \mathbb{Q}^d_p$
A point $x \in \mathbb{Q}_p^d$ is encoded by a sequence $(a_n)_{n \in \mathbb{Z}}$, $a_n \in \{0, 1, \ldots, p - 1\}^d$.

Let $0 \in \mathbb{Q}_p^d$ be the sequence with all digits equal to zero.
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**Caution! dangerous notation**
$a_n$ represents the local coordinates for a cube of $\mathbb{L}_{-n-1}$ inside a cube of $\mathbb{L}_{-n}$. 
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Moreover, rescaling is defined as follows. If \( x = (a_n)_{n \in \mathbb{Z}} \) then \( px := (a_{n-1})_{n \in \mathbb{Z}} \), i.e., upward shift.
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Likewise \( p^{-1}x \) is downward shift, and so on for the definition of \( p^kx, k \in \mathbb{Z} \).
Distance:

If \( x, y \in \mathbb{Q}^d \), define their distance as 
\[
| x - y |_p = p^k
\]
where \( k \) is the depth where the two paths merge.

Also let 
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| x |_p = | x - 0 |_p.
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Because of the dangerous notation 
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$$|p^k x|_p = p^{-1} |x|_p$$
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Lebesgue measure:

- Metric space $\mathbb{Q}_d$ with the Borel $\sigma$-algebra.
- Lebesgue measure $\mu$ which gives a volume $\mu(k)$ to closed balls of radius $k$.

Construction: take product of uniform probability measures on $(\{0, 1, \ldots, p-1\})^N$ for $B(0, 1)$. Do the same for the other closed unit balls, and collate.

The hierarchical unit lattice:
- Truncate the tree at level zero and take $L := L_0$. Using the identification of nodes with balls, define the hierarchical distance as $d(x, y) = \inf \{ |x - y| : x, y \in \mathbb{Q}_d \}$.
Lebesgue measure:

Metric space $\mathbb{Q}_p^d \rightarrow$ Borel $\sigma$-algebra $\rightarrow$ Lebesgue measure $d^d x$ which gives a volume $p^{dk}$ to closed balls of radius $p^k$. 
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$$d(x, y) = \inf\{|x - y|_p \mid x \in x, \ y \in y\}.$$
The massless Gaussian measure:

To every group of offsprings $G$ of a vertex $z \in L_{k+1}$ associate a centered Gaussian random vector $(\zeta_x)_{x \in G}$ with $p \times p$ covariance matrix made of $1-p-d$'s on the diagonal and $-p-d$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different groups are independent.

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The ancestor function: for $k < k'$, $x \in \mathbb{L}_k$, let $\text{anc}_{k'}(x)$ denote the ancestor in $\mathbb{L}_{k'}$. 

The massless Gaussian field $\phi(x), x \in \mathbb{Q}_{d_p}$ of scaling dimension $[\phi]$ is given by

$$
\phi(x) = \sum_{k \in \mathbb{Z}^{d_p}} [-k][\phi] \zeta_{\text{anc}_k}(x) \langle \phi(x) \phi(y) \rangle = c |x - y|^2[\phi]
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This is heuristic since $\phi$ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions.
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Test functions:

$f: \mathbb{Q}^d \to \mathbb{R}$ is smooth if it is locally constant. Define $S(\mathbb{Q}^d)$ as the space of compactly supported smooth functions. Take locally convex topology generated by the set of all semi-norms on $S(\mathbb{Q}^d)$.

Distributions: $S'(\mathbb{Q}^d)$ is the dual space with strong topology (happens to be same as weak-$\ast$). $S(\mathbb{Q}^d) \cong \bigoplus \mathbb{R}$, thus $S'(\mathbb{Q}^d) \cong \mathbb{R}^N$ with product topology. $S'(\mathbb{Q}^d)$ is a Polish space.
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Sample fields are true functions that are locally constant on scale $L^r$. These measures are scaled copies of each other.
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If the law of \( \phi(\cdot) \) is \( \mu_{C_0} \), then that of \( L^{-r[\phi]} \phi(L^r \cdot) \) is \( \mu_{C_r} \).
Fix the parameters $g, \mu$ and let $g_r = L^{-(3-4[\phi])} g$ and $\mu_r = L^{-(3-2[\phi])} \mu$. 
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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{ g_r : \phi^4 : c_r(x) + \mu_r : \phi^2 : c_r(x) \} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{Z_{r,s}} e^{-V_{r,s}(\phi)} d\mu c_r(\phi)$$
Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = (Z_2)^r \int_{\mathbb{Q}_p^3} \{ Y_2 : \phi_{r,s}^2 : c_r (x) - Y_0 L^{-2r[\phi]} \} j(x) \, d^3 x$$

for suitable parameters $Z_2, Y_0, Y_2$. 
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Our main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order). For the precise statement we need the approximate fixed point value

$$\bar{g}^* = \frac{p^\epsilon - 1}{36 L^\epsilon (1 - p^{-3})}$$
Theorems:

Theorem 1: A.A.-Chandra-Guadagni 2013

\[ \exists \rho > 0, \exists L_0, \forall L \geq L_0, \exists \epsilon_0 > 0, \forall \epsilon \in (0, \epsilon_0], \exists [\phi_2] > 2[\phi], \exists \text{fonctions } \mu(g), Y_0(g), Y_2(g) \text{ on } (\bar{g}^* - \rho \epsilon_0, \bar{g}^* + \rho \epsilon_0) \text{ such that if one lets } \mu = \mu(g), Y_0 = Y_0(g), Y_2 = Y_2(g) \text{ and } Z_2 = L - (\phi_2 - 2\phi), \text{ then the joint law of } \left( \phi_r, s, N_r[\phi_2], s \right) \text{ converge weakly and in the sense of moments to that of a pair } \left( \phi, N[\phi_2] \right) \text{ such that:}

1. \[ \forall k \in \mathbb{Z}, \left( L - k[\phi], L - k[\phi]^2 \right) = (\phi, N[\phi_2]) \]

2. \[ \langle \phi(1_{\mathbb{Z}^3}), \phi(1_{\mathbb{Z}^3}), \phi(1_{\mathbb{Z}^3}), \phi(1_{\mathbb{Z}^3}) \rangle_T < 0 \text{ i.e., } \phi \text{ is non-Gaussian. Here, } 1_{\mathbb{Z}^3} \text{ denotes the indicator function of } B(0, 1). \]

3. \[ \langle N[\phi_2](1_{\mathbb{Z}^3}), N[\phi_2](1_{\mathbb{Z}^3}) \rangle_T = 1. \]
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2. \[ \langle \phi(1\mathbb{Z}_p^3), \phi(1\mathbb{Z}_p^3), \phi(1\mathbb{Z}_p^3), \phi(1\mathbb{Z}_p^3) \rangle^T < 0 \text{ i.e., } \phi \text{ is non-Gaussian.} \text{ Here, } 1\mathbb{Z}_p^3 \text{ denotes the indicator function of } \overline{B}(0, 1). \]
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\[ \exists \text{fonctions } \mu(g), Y_0(g), Y_2(g) \text{ on } (\bar{g}_* - \rho \epsilon^2, \bar{g}_* + \rho \epsilon^2) \] such that if one lets \( \mu = \mu(g), Y_0 = Y_0(g), Y_2 = Y_2(g) \) and \( Z_2 = L^{-([\phi^2] - 2[\phi])} \) then the joint law of \((\phi_{r,s}, N_r[\phi^2_{r,s}])\) converge weakly and in the sense of moments to that of a pair \((\phi, N[\phi^2])\) such that:

1. \( \forall k \in \mathbb{Z}, (L^{-k[\phi]} \phi(L^k \cdot), L^{-k[\phi^2]} N[\phi^2](L^k \cdot)) \overset{d}{=} (\phi, N[\phi^2]). \)
2. \( \langle \phi(1_{\mathbb{Z}_p^3}), \phi(1_{\mathbb{Z}_p^3}), \phi(1_{\mathbb{Z}_p^3}), \phi(1_{\mathbb{Z}_p^3}) \rangle^T < 0 \) i.e., \( \phi \) is non-Gaussian. Here, \( 1_{\mathbb{Z}_p^3} \) denotes the indicator function of \( \overline{B}(0, 1) \).
3. \( \langle N[\phi^2](1_{\mathbb{Z}_p^3}), N[\phi^2](1_{\mathbb{Z}_p^3}) \rangle^T = 1. \)
The mixed correlation functions satisfy, in the sense of distributions,

\[
\langle \phi(L^{-k}x_1) \cdots \phi(L^{-k}x_n) N[\phi^2](L^{-k}y_1) \cdots N[\phi^2](L^{-k}y_m) \rangle = L^{-(n[\phi]+m[\phi^2])k} \langle \phi(x_1) \cdots \phi(x_n) N[\phi^2](y_1) \cdots N[\phi^2](y_m) \rangle
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Not too far, if one boldly extrapolates to $\epsilon = 1$, from the most precise available estimates concerning the short range 3D Ising model: $[\phi^2] - 2[\phi] = 0.376327 \ldots$ (JHEP 2016 by Kos, Poland, Simmons-Duffin and Vichi, using conformal bootstrap).
We also proved the law $\nu_{\phi \times \phi^2}$ of $(\phi, N[\phi^2])$, up to multiplying $\phi$ by a constant, is independent of $g$ in the interval $(\bar{g}_* - \rho \varepsilon^2, \bar{g}_* + \rho \varepsilon^2)$. 

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Theorem 2: A.A.-Chandra-Guadagni 2013

$\nu_{\phi\times\phi^2}$ is fully scale invariant, i.e., invariant under the action of the scaling group $p^\mathbb{Z}$ instead of the subgroup $L^\mathbb{Z}$. Moreover, $\mu(g)$ and $[\phi^2]$ are independent of the arbitrary factor $L$. 
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The two-point correlations are given in the sense of distributions by

$$\langle \phi(x)\phi(y) \rangle = \frac{c_1}{|x-y|^{2[\phi]}}$$

$$\langle N[\phi^2](x) N[\phi^2](y) \rangle = \frac{c_2}{|x-y|^{2[\phi^2]}}$$
Note that \( 2[\phi^2] = 3 - \frac{1}{3} \epsilon + o(\epsilon) \rightarrow \text{still } L^{1,\text{loc}} \! \)
Note that $2[φ^2] = 3 - \frac{1}{3}ε + o(ε) →$ still $L^{1,loc}$!

**Theorem 3: A.A., May 2015**

Use $ψ_i$ to denote the scaling limits $φ$ or $N[φ^2]$. Then, for all mixed correlation $∃$ a smooth function $⟨ψ_1(z_1)\cdotsψ_n(z_n)⟩$ on $(Q^3_p)^n\setminus{\text{Diag}}$ which is locally integrable (on the big diagonal Diag) and such that

$$E \, ψ_1(f_1)\cdotsψ_n(f_n) =$$

$$\int_{(Q^3_p)^n\setminus{\text{Diag}}} ⟨ψ_1(z_1)\cdotsψ_n(z_n)⟩ \, f_1(z_1)\cdots f_n(z_n) \, d^3z_1\cdots d^3z_n$$

for all test functions $f_1,\ldots,f_n \in S(Q^3_p)$. 
This hinges on showing the BNNFB (basic nearest neighbor factorized bound) of (A2016):

\[ | \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle | \leq O(1) \times \prod_{i=1}^{n} \frac{1}{|x_i - \text{n.n.}|[\psi_i]} \]

when \( z_1, \ldots, z_n \) are confined to a compact set.
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Hence, the emergent connection to the AdS/CFT correspondence.
1 Introduction

2 The hierarchical continuum

3 The rigorous hierarchical space-dependent renormalization group
The renormalization group idea in a nutshell:

- Want to study feature $Z(\vec{V})$ of some object $\vec{V} \in E$ but too hard!
- Find "simplifying" transformation $RG: E \rightarrow E$, such that $Z(RG(\vec{V})) = Z(\vec{V})$, and $\lim_{n \rightarrow \infty} RG^n(\vec{V}) = \vec{V}^\ast$ with $Z(\vec{V}^\ast)$ easy.

Example (Landen-Gauss): $\vec{V} = (a, b) \in E = (0, \infty)^2$

$$Z(\vec{V}) = \int_0^{\pi/2} d\theta \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$$

Take $RG(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$. 
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The renormalization group idea in a nutshell:

Want to study feature $\mathcal{Z}(\vec{V})$ of some object $\vec{V} \in \mathcal{E}$ but too hard!

Find “simplifying” transformation $\mathcal{R}G : \mathcal{E} \to \mathcal{E}$, such that $\mathcal{Z}(\mathcal{R}G(\vec{V})) = \mathcal{Z}(\vec{V})$, and $\lim_{n \to \infty} \mathcal{R}G^n(\vec{V}) = \vec{V}_*$ with $\mathcal{Z}(\vec{V}_*)$ easy.

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Take $\mathcal{R}G(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$. 
1st step: switch to unit lattice/cut-off

\[ S_{r,s}^T(f) := \log \mathbb{E}_{\nu_{r,s}} e^{i\phi(f)} = \log \]

\[ \int d\mu_{C_r}(\phi) \exp \left( - \int_{\Lambda_s} \{ g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r \} dx + \int \phi(x) f(x) dx \right) \]

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\[ = \log \frac{\int d\mu_{C_0}(\phi) I^{(r,r)}[f](\phi)}{\int d\mu_{C_0}(\phi) I^{(r,r)}[0](\phi)} \]
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\[ = \log \frac{\int d\mu_C(\phi) \mathcal{I}^{(r,r)}[f](\phi)}{\int d\mu_C(\phi) \mathcal{I}^{(r,r)}[0](\phi)} =: \log \frac{\mathcal{Z}(\mathcal{V}^{(r,r)}[f])}{\mathcal{Z}(\mathcal{V}^{(r,r)}[0])} \]
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\[
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\]

with

\[
\mathcal{I}^{(r,r)}[f](\phi) = \exp \left( - \int_{\Lambda_{s-r}} \{ g : \phi^4 : 0 \ (x) + \mu : \phi^2 : 0 \} d^3x \right. \\
\left. + L^{(3-\phi)} r \int \phi(x)f(L^{-r} x) d^3x \right)
\]
2nd step: define inhomogeneous RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Associated Gaussian measure is the law of the fluctuation field

$$\zeta(x) = \sum_{0 \leq k < \ell} p^{-k[\phi]} \zeta_{\text{anc}_k}(x)$$

$L$-blocks (closed balls of radius $L$) are independent. Hence
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\[
\int I^{(r,r)}[f](\phi) \; d\mu_{C_0}(\phi) = \int \int I^{(r,r)}[f](\zeta + \psi) \; d\mu_{\Gamma}(\zeta) d\mu_{C_1}(\psi)
\]

\[
= \int I^{(r,r+1)}[f](\phi) \; d\mu_{C_0}(\phi)
\]

with new integrand

\[
I^{(r,r+1)}[f](\phi) = \int I^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L \cdot)) \; d\mu_{\Gamma}(\zeta)
\]
Need to extract vacuum renormalization → better definition is

\[ \mathcal{I}^{(r,r+1)}[f](\phi) = e^{-\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]} \phi(L \cdot)) \, d\mu_\Gamma(\zeta) \]

so that

\[ \int \mathcal{I}^{(r,r)}[f](\phi) \, d\mu_{C_0}(\phi) = e^{\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r+1)}[f](\phi) \, d\mu_{C_0}(\phi) \]
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Repeat: \( I^{(r,r)} \rightarrow I^{(r,r+1)} \rightarrow I^{(r,r+2)} \rightarrow \ldots \rightarrow I^{(r,s)} \)
Need to extract vacuum renormalization → better definition is

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Repeat: \( \mathcal{I}^{(r,r)} \rightarrow \mathcal{I}^{(r,r+1)} \rightarrow \mathcal{I}^{(r,r+2)} \rightarrow \ldots \rightarrow \mathcal{I}^{(r,s)} \)

One must control

\[ S_T(f) = \lim_{r \to -\infty} \sum_{\substack{s \to \infty \quad r \leq q < s}} \left( \delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0]) \right) \]

limit of logarithms of characteristic functions.
Use a Brydges-Yau lift

\[ \vec{V}(r, q) \xrightarrow{\text{RG}_{\text{inhom}}} \vec{V}(r, q+1) \]

\[ I(r, q) \xrightarrow{\text{RG}_{\text{inhom}}} I(r, q+1) \]
Use a Brydges-Yau lift

\[ \mathbf{V}(r,q) \xrightarrow{RG_{inhom}} \mathbf{V}(r,q+1) \]

\[ \mathcal{I}(r,q) \xrightarrow{\downarrow} \mathcal{I}(r,q+1) \]

\[ \mathcal{I}(r,q)(\phi) = \prod_{\Delta \in L_0} \left[ e^{f \Delta \phi \Delta} \times \right. \]

\[ \left\{ \exp \left(-\beta_{4,\Delta} : \phi_\Delta^4 : c_0 - \beta_{3,\Delta} : \phi_\Delta^3 : c_0 - \beta_{2,\Delta} : \phi_\Delta^2 : c_0 - \beta_{1,\Delta} : \phi_\Delta^1 : c_0 \right) \right. \]

\[ \left. \times \left( 1 + W_{5,\Delta} : \phi_\Delta^5 : c_0 + W_{6,\Delta} : \phi_\Delta^6 : c_0 \right) \right. \]

\[ \left. + R_\Delta(\phi_\Delta) \right\} \]
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\[ I^{(r,q)}(\phi) = \prod_{\Delta \in \mathbb{L}_0} \left[ e^{f_{\Delta}\phi_{\Delta}} \times \{ \exp \left( -\beta_{4,\Delta} : \phi^4_{\Delta} : c_0 - \beta_{3,\Delta} : \phi^3_{\Delta} : c_0 - \beta_{2,\Delta} : \phi^2_{\Delta} : c_0 - \beta_{1,\Delta} : \phi^1_{\Delta} : c_0 \right) \times (1 + W_{5,\Delta} : \phi^5_{\Delta} : c_0 + W_{6,\Delta} : \phi^6_{\Delta} : c_0) \right] \]

\[ + R_{\Delta}(\phi_{\Delta}) \} \]

Dynamical variable is \( \vec{V} = (V_{\Delta})_{\Delta \in \mathbb{L}_0} \) with

\[ V_{\Delta} = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_{\Delta}, R_{\Delta}) \]
$R G_{\text{inhom}}$ acts on $E_{\text{inhom}}$, essentially,

$$\prod_{\Delta \in \mathcal{L}_0} \{ C^7 \times C^9(\mathbb{R}, \mathbb{C}) \}$$
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\prod_{\Delta \in L_0} \{ \mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C}) \}
\]

**Stable subspaces**

$E_{\text{hom}} \subset E_{\text{inhom}}$: spatially constant data.

$E \subset E_{\text{hom}}$: even potential, i.e., $g$, $\mu$’s only and $R$ even function.

Let $RG$ be induced action of $RG_{\text{inhom}}$ on $E$. 
3rd step: stabilize bulk (homogeneous) evolution

Show that $\forall q \in \mathbb{Z}$, $\lim_{r \to -\infty} \bar{V}(r,q)[0]$ exists, i.e.,

$$\lim_{r \to -\infty} RG^{q-r} \left( \bar{V}(r,r)[0] \right)$$

exists.
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exists.

$$RG \begin{cases} 
    g' &= L^\epsilon g - A_1 g^2 + \cdots \\
    \mu' &= L^{\frac{3+\epsilon}{2}} \mu - A_2 g^2 - A_3 g \mu + \cdots \\
    R' &= L^{(g, \mu)}(R) + \cdots 
\end{cases}$$

Tadpole graph with mass insertion

$A_3 = 12 L_3 - \frac{2}{5} \phi \int Q^3 p \Gamma(0, x) d^3 x$ is main culprit for anomalous scaling $\phi^2 - 2\phi > 0.$
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exists.

$$\begin{cases} 
  g' = L^\epsilon g - A_1 g^2 + \cdots \\
  \mu' = L^{\frac{3+\epsilon}{2}} \mu - A_2 g^2 - A_3 g \mu + \cdots \\
  R' = \mathcal{L}^{(g, \mu)}(R) + \cdots 
\end{cases}$$

Tadpole graph with mass insertion

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Irwin’s proof \(\rightarrow\) stable manifold \(W^s\)
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Restriction to \( W^s \) → contraction → IR fixed point \( v_* \).
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Construct unstable manifold $W^u$, intersect with $W^s$, transverse at $v_*$. 
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Restriction to $W^s$ → contraction → IR fixed point $v_*$.
Construct unstable manifold $W^u$, intersect with $W^s$, transverse at $v_*$.
Here, $\vec{V}^{(r,r)}[0]$ is independent of $r$: strict scaling limit of fixed model on unit lattice.
Irwin’s proof $\rightarrow$ stable manifold $W^s$
Restriction to $W^s$ $\rightarrow$ contraction $\rightarrow$ IR fixed point $v_*$. 
Construct unstable manifold $W^u$, intersect with $W^s$, transverse at $v_*$. 
Here, $\vec{V}^{(r,r)}[0]$ is independent of $r$: strict scaling limit of fixed model on unit lattice. (We can also do the Gaussian to non-Gaussian crossover continuum limit).
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\[ \forall q \in \mathbb{Z}, \lim_{r \to -\infty} \vec{V}^{(r,q)}[0] = v_* \]
Tangent spaces at fixed point: $E^s$ and $E^u$. $E^u = C e_u$, with $e_u$ eigenvector of $D_{v_*} RG$ for eigenvalue $\alpha_u = L^{3-2[\phi]} \times \mathbb{Z}_2 =: L^{3-[\phi^2]}$. 

4th step: control deviation from homogeneous evolution
\( \mathbf{V}(r,q)[f] - \mathbf{V}(r,q)[0] \), for all effective scale \( q \), uniformly in \( r \).
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1) \( \sum_{x \in G} \zeta_x = 0 \) a.s. \( \rightarrow \) deviation is 0 for \( q \) < local constancy scale of test function \( f \).

2) Deviation resides in closed unit ball containing origin for \( q \) > radius of support of \( f \) \( \rightarrow \) exponential decay for large \( q \).
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For source term with \( \phi^2 \) add

\[
Y_2 Z_2^r \int : \phi^2 : c_r (x) j(x) d^3x
\]

to potential. \( S_{r,s}'(f,j) \) now involves two test functions. After rescaling to unit lattice/cut-off

\[
Y_2 \alpha_u' \int : \phi^2 : c_0 (x) j(L^{-r} x) d^3x
\]

to be combined with \( \mu \) into \( (\beta_{2,\Delta})_{\Delta \in \mathbb{L}_0} \) space-dependent mass.
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$$\Psi(v, w) = \lim_{n \to \infty} RG^n(v + \alpha_u^{-n}w)$$

for $v \in W^s$ and all direction $w$ (especially $\int : \phi^2 :$).
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$\Psi(v, w)$ is holomorphic in $v$ and $w$.

This is essential for probabilistic interpretation of $\langle \phi, N[\phi^2] \rangle$ as pair of random variables in $S'(Q_p^3)$.
Thank you for your attention.