Space-dependent renormalization group and anomalous dimensions in a hierarchical model for 3d CFT

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Partly joint work with Ajay Chandra (Imperial)
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Recent developments in Constructive Field Theory
Columbia University, March 13, 2018

Main references:

(ACG2013) A.A., A. Chandra, G. Guadagni, “Rigorous quantum field theory functional integrals over the p-adics I: anomalous dimension”, arXiv 2013.

(A2013) A.A., “QFT, RG, and all that, for mathematicians, in eleven pages”, arXiv 2013.

(A2015) A.A., “Towards three-dimensional conformal probability”, arXiv 2015.

(A2016) A.A., “A second-quantized Kolmogorov-Chentsov theorem”, arXiv 2016.

1. Introduction
2. The hierarchical continuum
3. The rigorous hierarchical space-dependent renormalization group

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Used for generalizations of Zamolodchikov's c-"Theorem", study of scale vs. conformal invariance,...

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- Constructing explicit examples of holography or AdS/CFT correspondence.

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where $S[\phi]$ is an action for a field $\phi(x, x_{d+1})$ on AdS space and ϕ_{ext} makes it extremal for a boundary condition $\phi(x, x_{d+1}) \sim (x_{d+1})^{d-\Delta} j(x)$ when $x_{d+1} \to 0$.

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where m^2 is related to Δ and is allowed to be (not too) negative. This gives an expansion for connected CFT correlations in terms of tree-level Feynman diagrams (Witten diagrams). The simplest "Mercedes logo" 3-point Witten diagram reproduces the correct CFT prediction

$$\frac{O(1)}{|x_1 - x_2|^{\Delta_1+\Delta_2-\Delta_3} |x_1 - x_3|^{\Delta_1+\Delta_3-\Delta_2} |x_2 - x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

for $\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle$ by a calculation of Freedman, Mathur, Matusis and Rastelli 1999.

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The calculations of the last reference for scaling dimensions of Φ and Φ^2, for $N = 1$ in hierarchical case were made nonperturbatively rigorous in (ACG2013).

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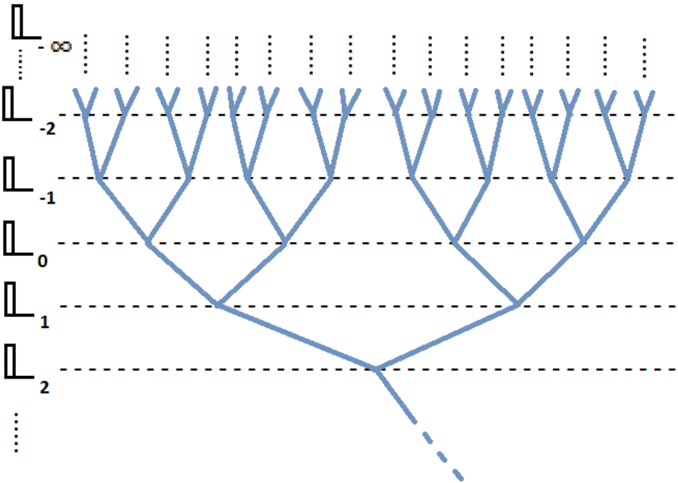
Hence $\mathbb{T} = \cup_{k \in \mathbb{Z}} \mathbb{L}_k$ naturally has the structure of a doubly infinite tree which is organized into layers or generations $\mathbb{L}_k$:



Picture for $d = 1$, $p = 2$

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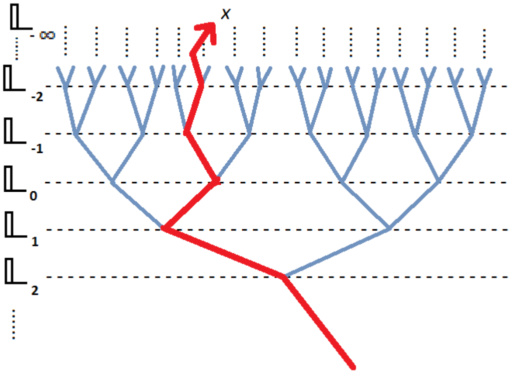
More precisely, these leafs at infinity are the infinite bottom-up paths in the tree.



A path representing an element $x \in \mathbb{Q}_p^d$

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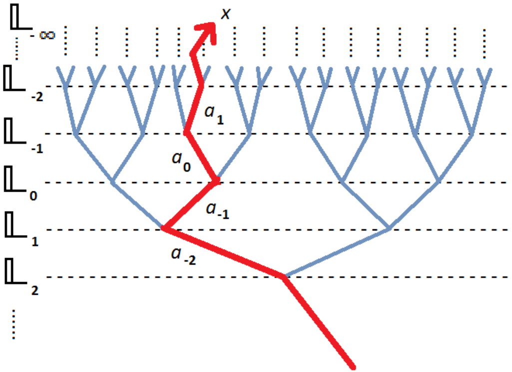

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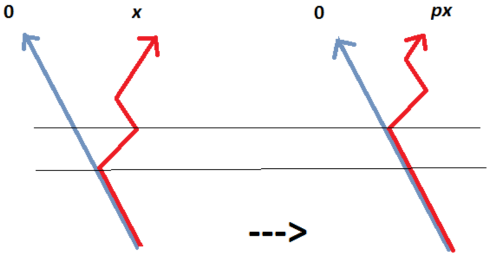

Likewise $p^{-1}x$ is downward shift, and so on for the definition of $p^k x$, $k \in \mathbb{Z}$.

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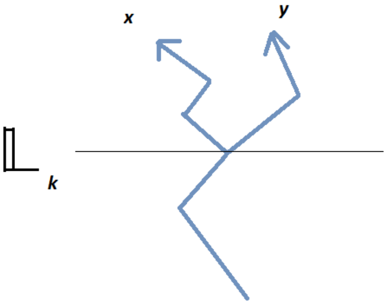

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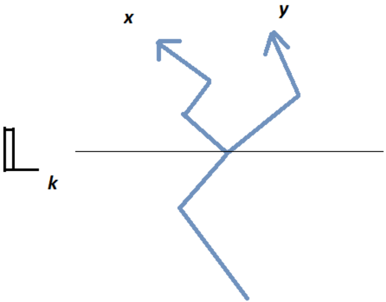

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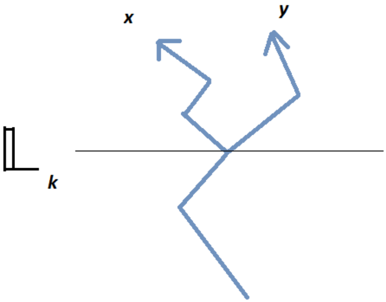

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$$|px|_p = p^{-1}|x|_p$$

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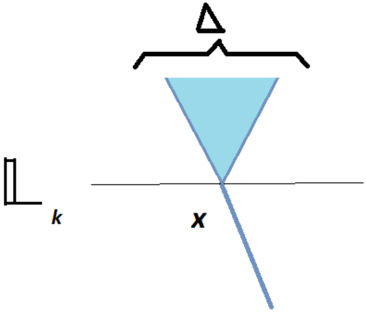

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The hierarchical unit lattice:

Truncate the tree at level zero and take $\mathbb{L} := \mathbb{L}_0$. Using the identification of nodes with balls, define the hierarchical distance as

$$d(\mathbf{x},\mathbf{y}) = \inf\{|x-y|_p \mid x \in \mathbf{x},\ y \in \mathbf{y}\} .$$

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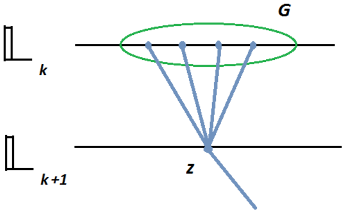

To every group of offsprings G of a vertex $\mathbf{z} \in \mathbb{L}_{k+1}$ associate a centered Gaussian random vector $(\zeta_{\mathbf{x}})_{\mathbf{x}\in G}$ with $p^d \times p^d$ covariance matrix made of $1 - p^{-d}$'s on the diagonal and $-p^{-d}$'s everywhere else. We impose that Gaussian vectors corresponding to different layers or different groups are independent.

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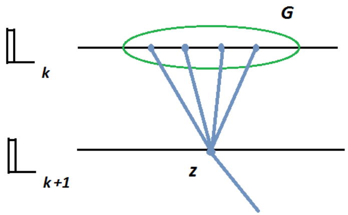

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$$\phi(x) = \sum_{k\in\mathbb{Z}} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

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This is heuristic since ϕ is not well-defined in a pointwise manner. We need random Schwartz(-Bruhat) distributions.

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If the law of $\phi(\cdot)$ is μ_{C_0}, then that of $L^{-r[\phi]}\phi(L^r \cdot)$ is μ_{C_r}.

Fix the parameters g, μ and let $g_r = L^{-(3-4[\phi])r} g$ and $\mu_r = L^{-(3-2[\phi])r} \mu$.

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Let

$$V_{r,s}(\phi) = \int_{\Lambda_s} \{g_r : \phi^4 :_{C_r} (x) + \mu_r : \phi^2 :_{C_r} (x)\} d^3x$$

and define the probability measure

$$d\nu_{r,s}(\phi) = \frac{1}{\mathcal{Z}_{r,s}} e^{-V_{r,s}(\phi)} d\mu_{C_r}(\phi)$$

Let $\phi_{r,s}$ be the random distribution in $S'(\mathbb{Q}_p^3)$ sampled according to $\nu_{r,s}$ and define the squared field $N_r[\phi_{r,s}^2]$ which is a deterministic function(al) of $\phi_{r,s}$, with values in $S'(\mathbb{Q}_p^3)$, given by

$$N_r[\phi_{r,s}^2](j) = (Z_2)^r \int_{\mathbb{Q}_p^3} \{Y_2 :\phi_{r,s}^2:_{C_r}(x) - Y_0 L^{-2r[\phi]}\}\, j(x)\, d^3x$$

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Our main result concerns the limit law of the pair $(\phi_{r,s}, N_r[\phi_{r,s}^2])$ in $S'(\mathbb{Q}_p^3) \times S'(\mathbb{Q}_p^3)$ when $r \to -\infty$, $s \to \infty$ (in any order).

For the precise statement we need the approximate fixed point value

$$\bar{g}_* = \frac{p^{\epsilon} - 1}{36 L^{\epsilon}(1 - p^{-3})}$$

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3. $\langle N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}), N[\phi^2](\mathbf{1}_{\mathbb{Z}_p^3}) \rangle^{\mathrm{T}} = 1$.

The mixed correlation functions satisfy, in the sense of distributions,

$$\langle \phi(L^{-k}x_1)\cdots\phi(L^{-k}x_n)N[\phi^2](L^{-k}y_1)\cdots N[\phi^2](L^{-k}y_m)\rangle$$

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Not too far, if one boldly extrapolates to $\epsilon=1$, from the most precise available estimates concerning the short range 3D Ising model: $[\phi^2]-2[\phi]=0.376327\ldots$ (JHEP 2016 by Kos, Poland, Simmons-Duffin and Vichi, using conformal bootstrap).

We also proved the law $\nu_{\phi\times\phi^2}$ of $(\phi, N[\phi^2])$, up to multiplying ϕ by a constant, is independent of g in the interval $(\bar{g}_* - \rho\epsilon^{\frac{3}{2}}, \bar{g}_* + \rho\epsilon^{\frac{3}{2}})$.

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The two-point correlations are given in the sense of distributions by

$$\langle\phi(x)\phi(y)\rangle = \frac{c_1}{|x-y|^{2[\phi]}}$$

$$\langle N[\phi^2](x)\ N[\phi^2](y)\rangle = \frac{c_2}{|x-y|^{2[\phi^2]}}$$

Note that $2[\phi^2] = 3 - \frac{1}{3}\epsilon + o(\epsilon) \to$ still $L^{1,\mathrm{loc}}$!

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Theorem 3: A.A., May 2015

Use ψ_i to denote the scaling limits ϕ or $N[\phi^2]$. Then, for all mixed correlation $\exists$ a smooth fonction $\langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle$ on $(\mathbb{Q}_p^3)^n \backslash \mathrm{Diag}$ which is locally integrable (on the big diagonal Diag) and such that

$$\mathbb{E}\ \psi_1(f_1) \cdots \psi_n(f_n) \ = \int_{(\mathbb{Q}_p^3)^n \backslash \mathrm{Diag}} \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle\ f_1(z_1) \cdots f_n(z_n)\ d^3 z_1 \cdots d^3 z_n$$

for all test functions $f_1, \ldots, f_n \in S(\mathbb{Q}_p^3)$.

This hinges on showing the BNNFB (basic nearest neighbor factorized bound) of (A2016):

$$| \langle \psi_1(z_1) \cdots \psi_n(z_n) \rangle | \leq O(1) \times \prod_{i=1}^{n} \frac{1}{|x_i - \mathrm{n.n.}|^{[\psi_i]}}$$

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Hence, the emergent connection to the AdS/CFT correspondence.

1. Introduction
2. The hierarchical continuum
3. The rigorous hierarchical space-dependent renormalization group

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Take $RG(a, b) = \left(\frac{a+b}{2}, \sqrt{ab}\right)$.

1st step: switch to unit lattice/cut-off

$$\mathcal{S}^{\mathrm{T}}_{r,s}(f) := \log \mathbb{E}_{\nu_{r,s}} e^{i\phi(f)} = \log$$

$$\frac{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r\}dx + \int \phi(x) f(x) dx\right)}{\int d\mu_{C_r}(\phi) \exp\left(-\int_{\Lambda_s}\{g_r : \phi^4 :_r (x) + \mu_r : \phi^2 :_r\}dx\right)}$$

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with

$$\mathcal{I}^{(r,r)}[f](\phi) = \exp\left(-\int_{\Lambda_{s-r}}\{g :\phi^4:_0(x) + \mu :\phi^2:_0\}d^3x\right.$$

$$\left.+L^{(3-[\phi])r}\int \phi(x)f(L^{-r}x)d^3x\right)$$

2nd step: define inhomogeneous RG

Fluctuation covariance $\Gamma := C_0 - C_1$.

Associated Gaussian measure is the law of the fluctuation field

$$\zeta(x) = \sum_{0 \le k < \ell} p^{-k[\phi]} \zeta_{\mathrm{anc}_k(x)}$$

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$$\int \mathcal{I}^{(r,r)}[f](\phi)\ d\mu_{C_0}(\phi) = \int\int \mathcal{I}^{(r,r)}[f](\zeta + \psi)\ d\mu_\Gamma(\zeta) d\mu_{C_1}(\psi)$$

$$= \int \mathcal{I}^{(r,r+1)}[f](\phi)\ d\mu_{C_0}(\phi)$$

with new integrand

$$\mathcal{I}^{(r,r+1)}[f](\phi) = \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

Need to extract vacuum renormalization $\rightarrow$ better definition is

$$\mathcal{I}^{(r,r+1)}[f](\phi) = e^{-\delta b(\mathcal{I}^{(r,r)}[f])} \int \mathcal{I}^{(r,r)}[f](\zeta + L^{-[\phi]}\phi(L\cdot))\ d\mu_\Gamma(\zeta)$$

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One must control

$$\mathcal{S}^{\mathrm{T}}(f) = \lim_{\substack{r\to-\infty \\ s\to\infty}} \sum_{r\le q<s} \left(\delta b(\mathcal{I}^{(r,q)}[f]) - \delta b(\mathcal{I}^{(r,q)}[0])\right)$$

limit of logarithms of characteristic functions.

Use a Brydges-Yau lift

$$\begin{tikzcd}
\vec{V}^{(r,q)} \arrow[r, "RG_{\text{inhom}}"] \arrow[d] & \vec{V}^{(r,q+1)} \arrow[d] \\
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\end{tikzcd}$$

Use a Brydges-Yau lift

$$\begin{array}{ccc} \vec{V}^{(r,q)} & \xrightarrow{RG_{\mathrm{inhom}}} & \vec{V}^{(r,q+1)} \\ \downarrow & & \downarrow \\ \mathcal{I}^{(r,q)} & \longrightarrow & \mathcal{I}^{(r,q+1)} \end{array}$$

$$\mathcal{I}^{(r,q)}(\phi) = \prod_{\substack{\Delta\in\mathbb{L}_0 \\ \Delta\subset\Lambda_{s-q}}} \Big[e^{f_\Delta \phi_\Delta} \times$$

$$\Big\{ \exp\left(-\beta_{4,\Delta} : \phi_\Delta^4 :_{C_0} -\beta_{3,\Delta} : \phi_\Delta^3 :_{C_0} -\beta_{2,\Delta} : \phi_\Delta^2 :_{C_0} -\beta_{1,\Delta} : \phi_\Delta^1 :_{C_0}\right)$$

$$\times \left(1 + W_{5,\Delta} : \phi_\Delta^5 :_{C_0} + W_{6,\Delta} : \phi_\Delta^6 :_{C_0}\right)$$

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Dynamical variable is $\vec{V} = (V_\Delta)_{\Delta\in\mathbb{L}_0}$ with

$$V_\Delta = (\beta_{4,\Delta}, \beta_{3,\Delta}, \beta_{2,\Delta}, \beta_{1,\Delta}, W_{5,\Delta}, W_{6,\Delta}, f_\Delta, R_\Delta)$$

RG_{inhom} acts on $\mathcal{E}_{\mathrm{inhom}}$, essentially,

$$\prod_{\Delta \in \mathbb{L}_0} \{\mathbb{C}^7 \times C^9(\mathbb{R}, \mathbb{C})\}$$

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Stable subspaces

$\mathcal{E}_{\rm hom}\subset\mathcal{E}_{\rm inhom}$: spatially constant data.
$\mathcal{E}\subset\mathcal{E}_{\rm hom}$: even potential, i.e., g, μ's only and R even function.
Let RG be induced action of $RG_{\rm inhom}$ on $\mathcal{E}$.

3rd step: stabilize bulk (homogeneous) evolution

Show that $\forall q \in \mathbb{Z}$, $\lim_{r\to-\infty} \vec{V}^{(r,q)}[0]$ exists, i.e.,

$$\lim_{r\to-\infty} RG^{q-r}\left(\vec{V}^{(r,r)}[0]\right)$$

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$$RG \left\{ \begin{array}{lllllllll} g' & = & L^{\epsilon} g & - & A_1 g^2 & + & \cdots & & \\ \mu' & = & L^{\frac{3+\epsilon}{2}} \mu & - & A_2 g^2 & - & A_3 g \mu & + & \cdots \\ R' & = & \mathcal{L}^{(g,\mu)}(R) & + & \cdots & & & & \end{array} \right.$$

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Tadpole graph with mass insertion

$$A_3 = 12 L^{3-2[\phi]} \int_{\mathbb{Q}_p^3} \Gamma(0,x)^2 \ d^3x$$

is main culprit for anomalous scaling $[\phi^2] - 2[\phi] > 0$.

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Thus

$$\forall q \in \mathbb{Z}, \qquad \lim_{r\to-\infty} \vec{V}^{(r,q)}[0] = v_*$$

Tangent spaces at fixed point: E^{s} and E^{u}.

$E^{\mathrm{u}} = \mathbb{C}e_{\mathrm{u}}$, with e_{u} eigenvector of $D_{v_*}RG$ for eigenvalue $\alpha_{\mathrm{u}} = L^{3-2[\phi]} \times Z_2 =: L^{3-[\phi^2]}$.

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For source term with ϕ^2 add

$$Y_2 Z_2^r \int :\phi^2:_{C_r}(x) j(x) d^3x$$

to potential. $\mathcal{S}^{\mathrm{T}}_{r,s}(f,j)$ now involves two test functions. After rescaling to unit lattice/cut-off

$$Y_2 \alpha_{\mathrm{u}}^r \int :\phi^2:_{C_0}(x) j(L^{-r}x) d^3x$$

to be combined with μ into $(\beta_{2,\Delta})_{\Delta\in\mathbb{L}_0}$ space-dependent mass.

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$\Psi(v,w)$ is holomorphic in v and w.

This is essential for probabilistic interpretation of $(\phi, N[\phi^2])$ as pair of random variables in $S'(\mathbb{Q}_p^3)$.

Thank you for your attention.