A HEEGAARD-FLOER INVARIANT OF BORDERED THREE-MANIFOLDS

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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

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Abstract

Heegaard-Floer homology, introduced around the turn of the millennium by P. Ozsváth and Z. Szabó, employs holomorphic curves to study 3- and 4-dimensional manifolds. The Heegaard-Floer package assigns to each 3-manifold Y a collection of abelian groups $\widehat{HF}(Y)$, $HF^+(Y)$, $HF^-(Y)$ and $HF^\infty(Y)$; to bordisms of three-manifolds, Heegaard-Floer homology assigns maps between these groups.

In this thesis, we extend the invariant $\widehat{HF}$ from closed three-manifolds to three-manifolds with arbitrary connected boundary. We assign to each 2-manifold S a differential algebra $\mathcal{A}(S)$, and to each 3-manifold with boundary parameterized by S a differential $\mathcal{A}(S)$-module. Our invariants generalize the invariants of knot complements constructed by Oszváth-Szabó and, independently, J. Rasmussen.

Acknowledgements

First, I would like to thank my advisor, Yasha Eliashberg, for teaching me much of the background for this thesis, suggesting the direction of research, and giving substantial help along the way. Both his mathematical and professional guidance have been invaluable. More generally, during my graduate career I have benefited from the excellent – and patient – teaching of most of the mathematics faculty at Stanford.

During my graduate career, much of my learning has been from discussions with other students, and with post docs. Those who helped me are too numerous to name, but Alex Ivrii and Lenny Ng particularly helped me to learn symplectic topology and Tyler Lawson to learn algebraic topology. Conversations with Dan Ramras over the last four years have helped me formulate precisely many of my ideas, and sort out the few good ones from the many bad.

More directly related to the material in this thesis, I would like to thank Peter Ozsváth for several helpful conversations and suggesting fruitful directions in which to develop the theory. I would also like to thank Joe Coffey, Lenny Ng and Zoltán Szabó for helpful conversations during the writing of this thesis. Benjamin Audoux for provided suggestions (not implemented here) about orientations, stylistic advice, and pointed out a mistake in a preliminary version of this thesis, for which I am grateful. Finally, several of the important ideas in this paper were suggested to me by Eaman Eftekhary and Matt Hedden, both of whom have thought about similar techniques. When possible, they are given credit more specifically in the text, but the effects of their suggestions are broader than specific citations make clear.

My first year as a graduate student was difficult for non-academic reasons. The support of my family, and also of Marc Schreiber and Ellen Eischen, was particularly important during that period, as well as before and after. Without their support, it is unlikely that this thesis would have been written.

Contents

Chapter 1

Introduction

In [OS04d], P. Ozsváth and Z. Szabó associated to an oriented 3-manifold Y, together with a $\mathrm{Spin}^{\mathbb{C}}$-structure s on Y, certain chain complexes $\widehat{\mathrm{CF}}(Y,s)$, $\mathrm{CF}^+(Y,s)$, $\mathrm{CF}^-(Y,s)$ and $\mathrm{CF}^\infty(Y,s)$, well-defined up to chain homotopy equivalence; the homology of these complexes, denoted $\widehat{\mathrm{HF}}(Y,s)$, $\mathrm{HF}^+(Y,s)$, $\mathrm{HF}^-(Y,s)$ and $\mathrm{HF}^\infty(Y,s)$ respectively, is called *Heegaard-Floer homology*. In [OSb], they associate to a smooth bordism W from Y_1 to Y_2, together with a $\mathrm{Spin}^{\mathbb{C}}$-structure $\mathfrak{t}$ on W, a map $F_{W,\mathfrak{t}} : \mathrm{HF}^*(Y_1, \mathfrak{t}|_{Y_1}) \to \mathrm{HF}^*(Y_2, \mathfrak{t}|_{Y_2})$, where $* \in \{\wedge, +, -, \infty\}$, making the Heegaard-Floer homology into some kind of "(3+1)-dimensional topological field theory." The Heegaard-Floer package also includes invariants of closed 4-manifolds ([OSb]), contact structures ([OS05a]), and knots ([OS04b], [Ras03]); we will discuss the last of these in somewhat more detail presently.

The Heegaard-Floer homology package contains a great deal of geometric information. For instance, the Heegaard-Floer groups of a 3-manifold Y detect the Thurston norm on $H_2(Y)$ ([OS04a]) and whether Y fibers over S^1 ([Nia]), and the 4-manifold invariant, which is believed to be equivalent to the Seiberg-Witten invariant, is strong enough to distinguish many smooth structures on 4-manifolds. Unfortunately, the Heegaard-Floer groups, and the maps between them, remain hard to compute.

One of the original motivations for Heegaard-Floer homology was to compute the Seiberg-Witten invariants of closed 4-manifolds by cutting them into simpler pieces and then reassembling the invariant of the closed manifold from certain invariants of the pieces. Similarly, one might hope to compute the Heegaard-Floer homology of closed 3-manifolds by cutting the closed manifolds into pieces and associating some invariants to the pieces; these invariants should contain enough information to reconstruct the Heegaard-Floer homology of the closed manifold, but still be computable, at least in some interesting examples.

In a special case, this program has already been carried out. In [OS04b] and [Ras03], Ozsváth and Szabó and, independently, J. Rasmussen, defined invariants of a knot K in a homology 3-sphere Y. We will focus on the case of $\widehat{\mathrm{CFK}}(Y, K)$, which takes the form of a filtered chain complex, well-defined up to filtered homotopy equivalence. (This fits into the framework discussed above by viewing $\widehat{\mathrm{CFK}}(Y, K)$ as associated to $Y \setminus K$, together with a framing of $\partial(Y \setminus K)$.) In [OSd], Ozsváth and Szabó show that the chain complex $\widehat{\mathrm{CFK}}(Y, K)$ contains enough information to reconstruct the groups $\widehat{\mathrm{HF}}(Y_r(K))$, where $Y_r(K)$ denotes r-surgery on K ($r \in \mathbb{Q}$); in our language, this can be thought of as gluing a solid torus to $\partial(Y \setminus K)$. Building on work of E. Eftekhary on longitude Floer homology and $\widehat{\mathrm{HFK}}$ of untwisted Whitehead doubles ([Eft05]), M. Hedden showed in [Hed05] and [Hed] that the knot invariant of cables on and twisted Whitehead doubles of K can be recovered from $\widehat{\mathrm{CFK}}(Y, K)$. In our language, this corresponds to gluing a manifold with two torus boundary components to $Y \setminus K$.

The knot invariant $\widehat{\mathrm{CFK}}(S^3, K)$ has been computed in many interesting examples. Indeed, remarkable recent work of C. Manolescu, Ozsváth, and S. Sarkar ([MOS]) provides an algorithm to compute $\widehat{\mathrm{CFK}}(S^3, K)$ for general K. The surgery formula of [OSd] is, therefore, a powerful tool for computing Heegaard-Floer groups of closed manifolds. It is also of use in answering abstract questions about surgeries; for instance, it can be used to give restrictions on which surgeries on knots yield Lens spaces ([OS05b]), and restrictions on cosmetic surgeries ([OSd], [Wan]).

In this paper, we present a generalization of $\widehat{\mathrm{CFK}}(Y, K)$ to manifolds with arbitrary connected, oriented boundary. To do so, in chapter 3 we associate a differential algebra $\mathcal{A}$ to each 2-manifold S. Then, in chapter 5, we associate to each 3-manifold Y with boundary parameterized by S a differential $\mathcal{A}$-module $\mathrm{CF}(Y)$, well-defined up to chain homotopy equivalence (over $\mathcal{A}$). We conjecture in Section 8.1 that given Y_1 and Y_2 with $\partial Y_1 = -\partial Y_2 = S$, one can reconstruct $\widehat{\mathrm{CF}}(Y_1 \cup_\partial Y_2)$ from $\mathrm{CF}(Y_1)$ and $\mathrm{CF}(Y_2)$. In the case that Y is the complement of a null-homologous knot, $\mathrm{CF}(Y)$ contains, and is closely related to $\widehat{\mathrm{CFK}}$ (Section 8.2); it is not known at present whether $\mathrm{CF}(Y)$ contains more information than $\widehat{\mathrm{CFK}}$ in this case.

Our construction falls short of our goals in two ways. The first is that, while we can compute the invariant $\mathrm{CF}(Y)$ in certain cases, it seems to be substantially harder to compute, in general, than $\widehat{\mathrm{CFK}}$. The second is that, for now, the gluing conjecture remains merely a conjecture.

Our construction involves studying holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$, where Σ is a special kind of Heegaard diagram, which we call a Heegaard diagram with boundary, for a manifold with boundary. (In the closed case, this reduces to the cylindrical formulation of Heegaard-Floer homology given in [Lip].) Rather than being closed, the surface Σ has a single puncture, so the space $\Sigma \times [0,1] \times \mathbb{R}$ has two kinds of infinities, $\pm\infty$ (in $\mathbb{R}$) and "east ∞" (in Σ).

At east ∞, holomorphic curves are asymptotic to certain Reeb chords, at certain heights (in $\mathbb{R}$). The $\mathbb{R}$-coordinate gives a (partial) order on the Reeb chords at east ∞ to which a curve is asymptotic; the algebra $\mathcal{A}$ keeps track of this partial order. In codimension 1, the $\mathbb{R}$-coordinates of Reeb chords can come together. Also, a Reeb chord which is the concatenation of two other Reeb chords can split apart. These phenomena are recorded in the differential on $\mathcal{A}$.

With the correct definition of $\mathcal{A}$, it is relatively easy to define the Heegaard-Floer differential module: it is generated, over $\mathcal{A}$, by the obvious analogs of generators of ordinary Heegaard-Floer homology. The differential counts rigid holomorphic curves. The asymptotics are tracked by coefficients in $\mathcal{A}$. One must then prove that, up to chain homotopy equivalence, the Heegaard-Floer module is independent of the choice of Heegaard diagram with boundary. This is done in chapters 6 and 7.

We stress that the invariant $\mathrm{CF}(Y)$ depends on not just Y but also the parameterization of ∂Y by a fixed reference surface. (In fact, $\mathrm{CF}(Y)$ depends on the reference surface S only through a Morse function which we pull back from S to ∂Y.) So, for instance, there are many different invariants associated to a solid torus, depending on a choice of framing for the boundary. (This is, of course, the case for $\widehat{\mathrm{CFK}}$ as well, which depends on a choice of meridian for K.) One can not hope to do much better: to specify a gluing of two 3-manifolds, one needs to specify an identification of their boundaries; different identifications lead to different glued manifolds.

Chapter 2

The topology of Heegaard diagrams with boundary

2.1 Heegaard diagrams for 2-manifolds

Fix a closed, connected, orientable 2-manifold B of genus k. (B stands for "boundary".) Let f be a self-indexing Morse function on B with unique index 0 and 2 critical points. Let C denote the circle $f^{-1}(1/2)$, and $\{b_i^\pm\}_{i=1}^{2k}$ denote the $2k$ descending spheres of index 1 critical points in C. (That is, each $b_i^\pm$ is a pair of points, which is the intersection of the ascending disk of some index 1 critical point of f with C.) We will call the data $(C, \{b_i^\pm\})$ a *Heegaard diagram for* B. Fix also a point $\mathfrak{z} \in C \setminus \left(\bigcup_{i=1}^{2k}\{b_i^\pm\}\right)$. The data $(C, \{b_i^\pm\}, \mathfrak{z})$ is a *pointed Heegaard diagram for* B. Note that the basepoint $\mathfrak{z}$ allows us to order the points $b_i^\pm$ canonically by starting at $\mathfrak{z}$ and then reading them off counter-clockwise.

We can recover the original surface B from the data $(C, \{b_i^\pm\})$ as follows. Thicken C to $C\times[0,1]$. Attach a 1-handle (thickened interval) to each pair of points $b_i^\pm\times\{1\}$. There is a unique way to do this so that the result is an orientable surface (with boundary). If $(C, \{b_i^\pm\})$ was the Heegaard diagram of some closed orientable surface then after attaching these handles the result has two boundary components. Cap each with a disk.

This construction also provides an explicit description for $H_1(B)$. Each pair $b_i^\pm$

gives a generator for $H_1(B)$ by gluing the core of the 1-handle corresponding to $b_i^{\pm}$ with the arc in $C\backslash\{\mathfrak{z}\}$ between $b_i^{\pm}$. Alternately, for a pointed Heegaard diagram we can view $H_1(B)$ as being generated by a certain subgroup of $H_1(C, \bigcup\{b_i^{\pm}\})$: the subgroup generated by arcs connecting b_i^+ to b_i^- not covering $\mathfrak{z}$.

2.2 Morse functions on 3-manifolds with boundary

Fix a 3-manifold with boundary Y and self-indexing Morse function f on Y such that

1. The restriction $(df)|_{\nu(\partial Y)}$ of df to the normal bundle to the boundary is zero.
2. The function f has a unique index 3 critical point. This critical point lies on ∂Y. It is also the unique index 2 critical point of $f|_{\partial Y}$.
3. The function f has a unique index 0 critical point. This critical point lies on ∂Y. It is also the unique index 0 critical point of $f|_{\partial Y}$.
4. Every index 1 critical point of $f|_{\partial Y}$ is an index 2 critical point of f.

Notice that it follows that the gradient flow of f preserves ∂Y, and has no index 1 critical points on ∂Y. There may be more index 2 critical points of f, as well as index 1 critical points, inside Y. We show in Lemma 2.2.1 below that such an f always exists.

In this thesis we will first construct invariants of the pair $(Y, f|_{\partial Y})$ which are unchanged under deformations of $f|_{\partial Y}$ through Morse functions (as well as any changes of f in the interior of Y).

We call $f^{-1}(3/2)$ the Heegaard surface corresponding to f, and denote it Σ_g, where g is the genus of $f^{-1}(3/2)$. This is a surface with a single boundary component. Let $\alpha_1, \cdots, \alpha_g$ denote the intersection of the ascending disks of index 1 critical points of f with Σ_g, and $\beta_1, \cdots, \beta_\ell$ the intersections of the descending disks of index 2 critical points of f with Σ_g. Order the β_i so that $\beta_1, \cdots, \beta_{2k}$ are arcs and $\beta_{2k+1}, \cdots, \beta_\ell$ are circles.

From the data $(\Sigma, \vec{\alpha} = \{\alpha_1, \cdots, \alpha_g\}, \vec{\beta} = \{\beta_1, \cdots, \beta_\ell\})$, which we call a *Heegaard diagram with boundary*, we can recover $(Y, \partial Y)$ as follows. Thicken Σ to $\Sigma \times [0,1]$. Recall (Section 2.1) that $(\partial\Sigma, \{\beta_i \cap \partial\Sigma\})$ specifies a surface B, which contains by construction a copy of $\partial\Sigma \times [0,1]$. We now plumb together $\Sigma \times [0,1]$ and $B \times [0,1]$. That is, identify a collar neighborhood of $\partial\Sigma \subset \Sigma$ with this $\partial\Sigma \times [0,1] \subset B \times [0,1]$. Glue a thickened copy $B \times [0,1]$ of B to $\Sigma \times [0,1]$ by identifying $(p,s,r) \in \partial\Sigma \times [0,1] \times [0,1] \subset B \times [0,1]$ with $(p,r,s) \in \partial\Sigma \times [0,1] \times [0,1] \subset \Sigma \times [0,1]$. Call the result Y_0. The arcs $\beta_i \times \{1\}$ extend to pairwise disjoint simple closed curves $\overline{\beta}_i$ in Y_0 by joining them with the cores of the 1-handles in B. Now glue 2-handles (thickened disks) along the $\alpha_i \times \{0\}$ and the $\overline{\beta}_i$. Call this new space Y_1. It is a (topological) manifold with boundary. The manifold Y_1 has two boundary components, each homeomorphic to S^2, and one boundary component homeomorphic to B. Fill the S^2 boundary components with three-balls. The result is homeomorphic to the original manifold Y. See Figure 2.2.

Observe that this construction gives a recipe for obtaining the homology of Y from the Heegaard diagram. That is,

$$\begin{aligned}
H_1(Y) &= \operatorname{coker}\left((\oplus H_1(\alpha_i)) \oplus \left(\oplus H_1(\overline{\beta}_j)\right) \to H_1(\Sigma \cup_\partial B)\right) \\
H_2(Y) &= \ker\left((\oplus H_1(\alpha_i)) \oplus \left(\oplus H_1(\overline{\beta}_j)\right) \to H_1(\Sigma \cup_\partial B)\right) \\
H_1(Y, \partial Y) = H_1(Y/\partial Y) &= \operatorname{coker}\left((\oplus H_1(\alpha_i)) \oplus (\oplus H_1(\beta_j)) \to H_1(\Sigma/\partial\Sigma)\right) \\
H_2(Y, \partial Y) = H_2(Y/\partial Y) &= \ker\left((\oplus H_1(\alpha_i)) \oplus (\oplus H_1(\beta_j)) \to H_1(\Sigma/\partial\Sigma)\right)
\end{aligned}$$

Note that there are restrictions on which β-curves are possible. In particular, if there are $2k$ β-arcs, ℓ β-circles, and g α-circles then $2k + 2\ell = 2g$. This follows by considering the closed 3-manifold obtained by doubling Y along ∂Y.

Another restriction is that the images of the β-curves in $H_1(\Sigma, \partial\Sigma)$ are linearly independent. If they weren't, in the induced Heegaard diagram for the double $Y \cup_\partial -Y$ there would be a linear dependence between the β-circles.

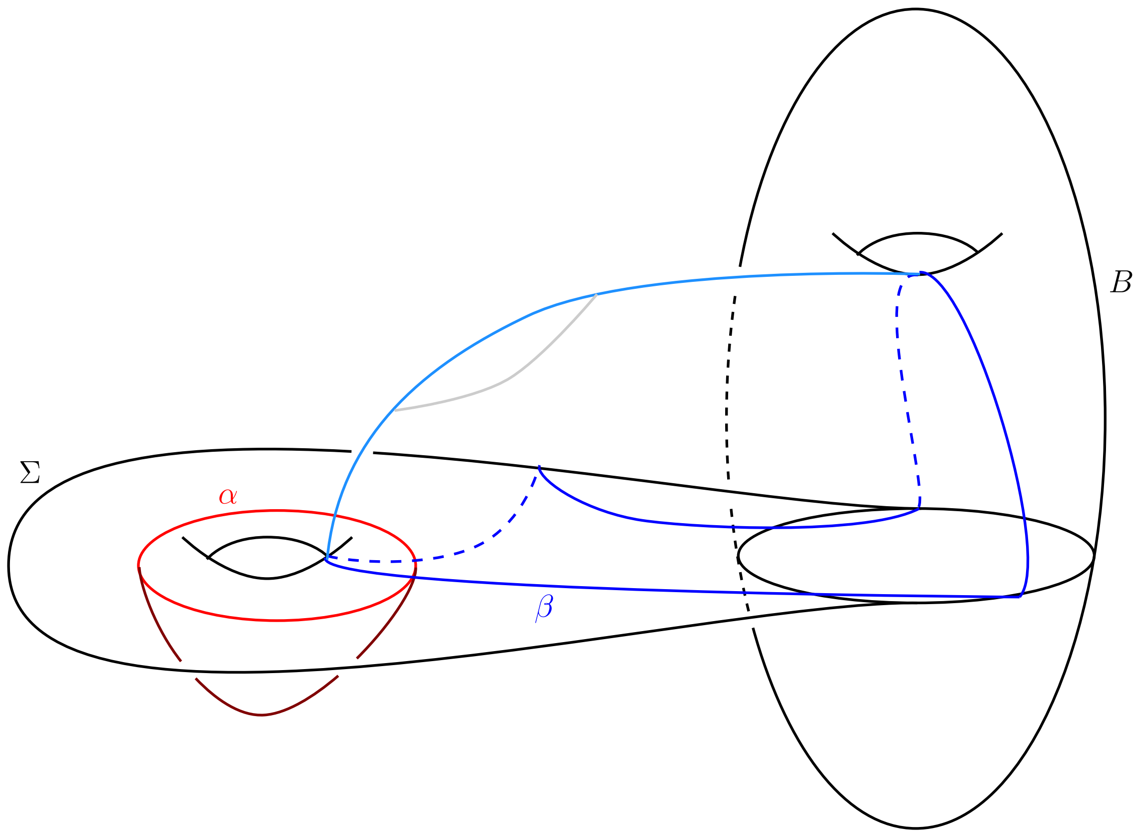


Figure 2.1: Constructing a manifold with boundary from a bordered Heegaard diagram

That these are the only restrictions is clear, by the doubling construction.

Lemma 2.2.1 *Every orientable* 3*-manifold* Y *with connected boundary admits a Heegaard diagram with boundary.*

Proof One can see this from the Morse-theoretic description as follows. Choose a Morse function f in a neighborhood of ∂Y with the desired properties; this is clearly possible. Extend f to a Morse function on Y arbitrarily. We need to eliminate all index 0 and 3 critical points of f in the interior of Y. Since Y is connected, for each index 0 critical point a in the interior of Y there is an index 1 critical point b in Y so that there is a single flow line between a and b. By a standard handle cancellation lemma (e.g., [Mil65, Theorem 5.4], one can cancel a against b. Reversing f, the same argument applies to cancel any excess index 3 critical point d. In this case, we need to check that the index 2 critical point c used to cancel d is contained in the interior of Y. This follows from the fact that the ascending sphere of an index 2 critical point of f in ∂Y is contained entirely in ∂Y. □

In Section 2.4 we will discuss how to construct Heegaard diagrams with boundary in practice.

Lemma 2.2.2 *Suppose that f_1 and f_2 are Morse functions on Y inducing Heegaard diagrams with boundary $(\Sigma^1, \vec{\alpha}^1, \vec{\beta}^1)$ and $(\Sigma^2, \vec{\alpha}^2, \vec{\beta}^2)$ for Y, and that the restrictions of f_1 and f_2 to a collar neighborhood of ∂Y agree. Then $(\Sigma^1, \vec{\alpha}^1, \vec{\beta}^1)$ and $(\Sigma^2, \vec{\alpha}^2, \vec{\beta}^2)$ are related by a sequence of the following moves.*

- *Isotopies of the α- and β-curves, fixed near $\partial\Sigma$,*
- *Handleslides between α-circles,*
- *Handleslides between β-circles,*
- *Handleslides of β-arcs over β-circles,*
- *Stabilizations (taking the connect sum with a standard genus 1 Heegaard diagram for S^3) and destabilizations (the inverse operation).*

Further, for given $\mathfrak{z}^i \in \Sigma^i \setminus (\vec{\alpha}^i \cup \vec{\beta}^i)$, we can assume all stabilizations occur in the connected component of $\Sigma^1 \setminus (\vec{\alpha}^1 \cup \vec{\beta}^1)$ containing $\mathfrak{z}^1$, prior to any isotopies and handleslides, and all destabilizations occur in the connected component of $\Sigma^2 \setminus (\vec{\alpha}^2 \cup \vec{\beta}^2)$ containing $\mathfrak{z}^2$, after all isotopies and handleslides.

Proof The proof, which is standard handle calculus, is the same as the proof of Proposition 2.2 in [OS04d]. (One might be concerned that handleslides of β-circles over β-arcs, whatever that would mean, could be necessary, but this is prohibited by the fact that the ascending sphere of each index 2 critical point of f in ∂Y is entirely contained in ∂Y.)

The second part of the statement is clear: performing a stabilization in any other component of $\Sigma^1 \backslash (\vec{\alpha}^1 \cup \vec{\beta}^1)$ is the same as performing a stabilization in the component containing $\mathfrak{z}^1$ and then a sequence of handleslides, passing α- and β-curves over the new handle; similarly, one can trade an isotopy (or handleslide) followed by a stabilization for a stabilization followed by isotopies and handleslides. (Compare [OSb, Lemma 2.10].) □

2.3 Notation

We will generally denote a Heegaard diagram with boundary by $(\Sigma, \vec{\alpha}, \vec{\beta})$ where the genus of Σ is g, $\vec{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$ are g circles, and $\vec{\beta} = \{\beta_1, \cdots, \beta_{2k}, \beta_{2k+1}, \cdots, \beta_{g+k}\}$ where $\beta_1, \cdots, \beta_{2k}$ are arcs and $\beta_{2k+1}, \cdots, \beta_{g+k}$ are circles. Let $C = \partial\Sigma$ and $\boldsymbol{b} = \vec{\beta} \cap C$. Our Heegaard diagrams will always be *pointed*, i.e., come with a choice of distinguished basepoint $\mathfrak{z}$ in $\mathrm{int}(\Sigma)$ so that $\mathfrak{z}$ can be connected to $\partial\Sigma$ by an arc in $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$. We will sometimes think of $\mathfrak{z}$ as a point in $\partial\Sigma$ (in the same connected component of $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$). The invariants will depend only on the connected component of $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$ containing $\mathfrak{z}$, not on $\mathfrak{z}$ itself.

2.4 Examples

Figure 2.4 shows a Heegaard diagram with boundary for a solid torus. Note that if we glue two copies of this Heegaard diagram along their common boundary after performing a half twist, we obtain a Heegaard diagram for S^3.

This is not, in fact, the only genus one Heegaard diagram with boundary for a solid torus. Others are given by taking the α-circle to be any other essential curve in the the torus, disjoint from the boundary. See, for example, Figures 5.1 – 5.3.

Another interesting class of examples is knot complements. E. Eftekhary and, independently, M. Hedden pointed out that a Heegaard diagram with boundary of $S^3 \backslash K$ can be constructed as follows. Let $(\Sigma, \{\alpha_1, \cdots, \alpha_g\}, \{\beta_1, \cdots, \beta_{g-1}\})$ denote an ordinary Heegaard diagram for $S^3 \setminus K$. Let $\lambda \subset \Sigma$ be a longitude of K, and $\mu \subset \Sigma$ a meridian of K, and assume that λ and μ intersect in a single point. Obtain Σ' from

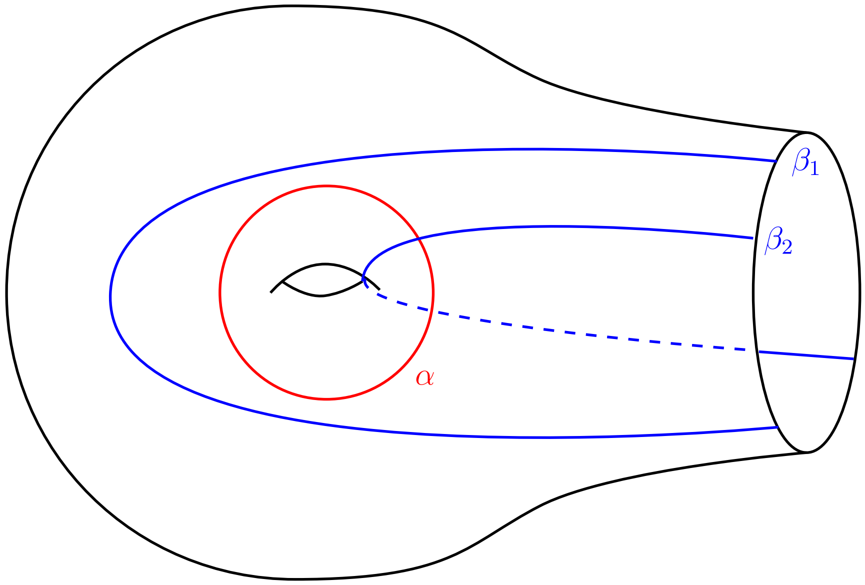
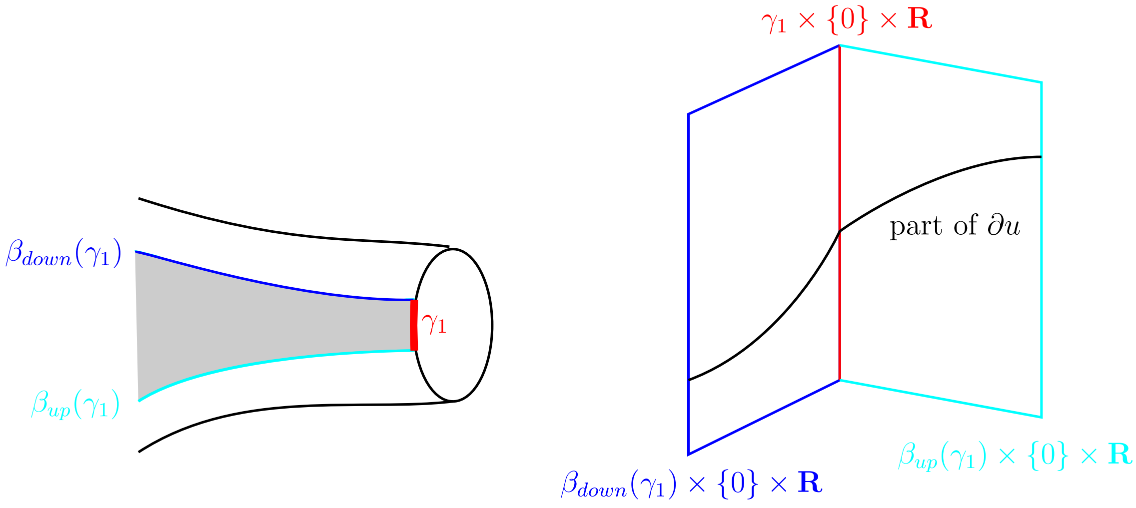


Figure 2.2: Heegaard diagram for a solid torus

Σ by deleting a small disk around $\lambda \cap \mu$. Then $(\Sigma', \{\alpha_1, \cdots, \alpha_g\}, \{\beta_1, \cdots \beta_{g-1}, \lambda, \mu\})$ is a Heegaard diagram with boundary for $S^3 \setminus K$. (This generalizes in an obvious way to complements of knots in other manifolds.)

More generally, suppose that one is given a relative handle decomposition of $(Y, \partial Y)$ (with a single 3-handle and no 1-handles). Fix a point q in ∂Y and embedded simple closed curves in $\pi_1(\partial Y, q)$ giving a π_1-framing for ∂Y and disjoint except at q. Let Σ_0 be a surface with genus $g(\partial Y)$ and fix $p \in \Sigma_0$. Choose simple closed curves $\beta_1, \cdots, \beta_{2k} \in \pi_1(\partial Y, q)$, $k = g(\partial Y)$, giving a π_1-framing of Σ_0, disjoint except at q. Then the choice of the β_i together with the π_1-framing of Σ_0 identifies Σ_0 with ∂Y (up to isotopy).

We are given a description of Y by attaching 1- and 2-handles, and a single 3-handle, to $\partial Y \cong \Sigma_0$. For each 1-handle, attach a cylinder in the corresponding way to Σ_0, and let β_{k+i} be a belt circle for the 1-handle (in the cylinder). Call the result of gluing the cylinders for the 1-handles $(\Sigma_1, \beta_1, \cdots, \beta_{k+m})$. Next, let α_i be an attaching circle for the i^{th} 2-handle. Then $(\Sigma_1, \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_{k+m})$ is a Heegaard diagram with boundary for Y.



(a) Picture in Σ. (b) Picture in $\vec{\beta} \times \{0\} \times R$.

Figure 2.3: (a) The definition of $\beta_{\text{up}}(\gamma)$ and $\beta_{\text{down}}(\gamma)$. Shaded region shows the image of $\pi_\Sigma \circ u$ for u a curve converging to γ_1. (b) Motivation for the terminology. The curve is part of the intersection of $u(\partial S)$ with the β-cylinders. The vertical line in the center is east ∞.

2.5 Reeb Chords

By a *Reeb chord at east* ∞ we mean an arc in $\partial\Sigma\backslash\{\mathfrak{z}\}$ with endpoints on $\boldsymbol{b}$. We label the Reeb chords at east ∞ by $\gamma_1, \cdots, \gamma_N$. Given a Reeb chord γ at east infinity we can define $\beta_{down}(\gamma)$ and $\beta_{up}(\gamma)$ of γ as in Figures 2.3.

Given a Reeb chord γ_i at east infinity, we write $\gamma_i = \gamma_j \uplus \gamma_k$ if γ_j and γ_k intersect only at one endpoint, $\gamma_i = \gamma_j \cup \gamma_k$, and γ_j is north (counterclockwise with respect to the "outward normal first" orientation) of γ_k, i.e., $\beta_{\text{up}}(\gamma_j) = \beta_{\text{down}}(\gamma_k)$. See Figures 2.3 and 2.4. We will say that γ_i *runs from* $\beta_{\text{down}}(\gamma_i)$ *to* $\beta_{\text{up}}(\gamma_i)$. We will say γ_i *runs between* β_j *and* β_k if the endpoints of γ_i are in $(\beta_j \cap \partial\Sigma) \cup (\beta_k \cap \partial\Sigma)$. We will sometimes drop the word "runs" from "runs between" or "runs from".

Note that since each β-arc intersects $\partial\Sigma$ in two points, it is possible to have, for instance, $\beta_{\text{up}}(\gamma_i) = \beta_{\text{down}}(\gamma_i)$ (i.e., γ_i runs from β_1 to β_1) and other similarly confusing phenomena. See, e.g., Figure 3.1.

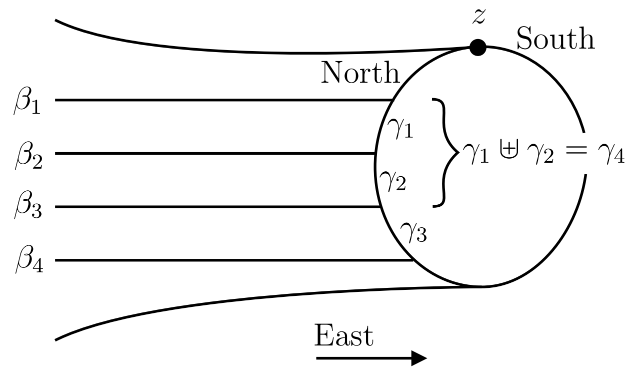


Figure 2.4: Example of $\uplus$.

2.6 Maps, homology classes and $\mathrm{Spin}^{\mathbb{C}}$-structures

Most of the preliminaries discussed in this section have analogs in traditional Heegaard-Floer homology; see [Lip, Sections 1 and 2].

We will define Heegaard-Floer homology by studying curves in $W = \Sigma \times [0,1] \times \mathbb{R}$. Viewing $\partial\Sigma$ as a puncture p, W has three kinds of infinity: $\Sigma \times [0,1] \times \{+\infty\}$, which we call $+\infty$, $\Sigma \times [0,1] \times \{-\infty\}$, which we call $-\infty$, and $p \times [0,1] \times \mathbb{R}$, which we call *east* ∞. Let $C_\alpha = \vec{\alpha} \times \{1\} \times \mathbb{R}$ and $C_\beta = \{\vec{\beta} \times \{0\} \times \mathbb{R}\}$. There are projection maps $\pi_\Sigma : W \to \Sigma$ and $\pi_\mathbb{D} : W \to [0,1] \times \mathbb{R}$.

By an *intersection point* we mean a g-tuple of points $\vec{x} = \{x_i \in \alpha_i \cap \beta_{\sigma(i)}\}_{i=1}^{g}$ so that exactly one x_i lies on each α-circle, exactly one x_i lies on each β-circle, and at most one x_i lies on each β-arc.[1] An intersection point specifies a g-tuple of arcs $\{x_i \times [0,1]\}$ in $\Sigma \times [0,1] \times \{\pm\infty\}$; with respect to a split symplectic form ω on $\Sigma \times [0,1] \times \mathbb{R}$, such arcs are leaves of the characteristic foliation on $\Sigma \times [0,1] \times \{\pm\infty\}$, with endpoints on $C_\alpha \cup C_\beta$. Sometimes when we write $\vec{x}$ we will in fact mean the corresponding g-tuple of leaves; the meaning should be clear in context.

Fix intersection points $\vec{x}$ and $\vec{y}$, and Reeb chords $\gamma_{i_1}, \cdots, \gamma_{i_k}$ at east ∞. We will be interested in holomorphic maps $u : (S, \partial S) \to (W, C_\alpha \cup C_\beta)$, with respect to an almost complex structure J on W satisfying certain properties explained in chapter 4.1, asymptotic to $\vec{x}$ (or rather, $\vec{x} \times [0,1]$) at $-\infty$, $\vec{y}$ at $+\infty$ and $\{\gamma_{i_j} \times \{0\} \times \{t_j\}\}$

[1] The term comes from thinking of $\vec{x}$ as an intersection point between certain tori in $\mathrm{Sym}^g(\Sigma)$.

at east ∞, and disjoint from $\{\mathfrak{z}\} \times [0,1] \times \mathbb{R}$. We say such curves *connect $\vec{x}$ to $\vec{y}$*. The space of curves connecting $\vec{x}$ to $\vec{y}$ breaks into homology classes in an obvious way. Let $\pi_2(\vec{x}, \vec{y})$ denote the set of homology classes of curves connecting $\vec{x}$ to $\vec{y}$. (The use of π_2 to denote homology classes is a holdover from˜[OS04d], in which the corresponding objects are homotopy classes of disks in $\mathrm{Sym}^g(\Sigma)$.)

Let $\Sigma^\circ = (\Sigma/\partial\Sigma)\setminus\mathfrak{z}$. Observe that

$$\pi_2(\vec{x}, \vec{x}) = H_2\left(\Sigma^\circ \times [0,1], (\vec{\alpha} \times \{1\}) \cup (\vec{\beta} \times \{0\})\right).$$

There are concatenation operations $\pi_2(\vec{x}, \vec{y}) \times \pi_2(\vec{y}, \vec{z}) \to \pi_2(\vec{x}, \vec{z})$ and inversions $\pi_2(\vec{x}, \vec{y}) \to \pi_2(\vec{y}, \vec{x})$. If $\pi_2(\vec{x}, \vec{y})$ is nonempty, concatenation of $\pi_2(\vec{x}, \vec{x})$ with any given element of $\pi_2(\vec{x}, \vec{y})$ gives a bijection $\pi_2(\vec{x}, \vec{x}) \to \pi_2(\vec{x}, \vec{y})$.

Given $w \in \Sigma\setminus(\vec{\alpha} \cup \vec{\beta})$ and a (topological) representative u of an element $A \in \pi_2(\vec{x}, \vec{y})$, write $n_w(u)$ to mean the local multiplicity of $\pi_\Sigma \circ u$ at w. The number $n_w(u)$ does not depend on u, so we may write $n_w(A)$ to mean $n_w(u)$ for some representative u of A. Note that if w and w' are in the same component of $\Sigma\setminus(\vec{\alpha} \cup \vec{\beta})$ then $n_w(A) = n_{w'}(A)$. Our requirement that curves be disjoint from $\{\mathfrak{z}\} \times [0,1] \times \mathbb{R}$ means that $n_{\mathfrak{z}}(A) = 0$ for any homology class A.

By a *cellular 2-chain in* Σ we mean a formal linear combination (with integer coefficients) of connected components of $\Sigma\setminus(\vec{\alpha} \cup \vec{\beta})$. (Note that this is a slight abuse of terminology, since some components may not be disks.) Given an element $A \in \pi_2(\vec{x}, \vec{y})$ there is a corresponding cellular chain in Σ, called the *domain of* A, where the coefficient of a component D is $n_w(A)$ for some point $w \in D$. The concatenation operations on π_2 correspond to addition of domains, and inversion corresponds to negation.

Lemma 2.6.1 *There is a natural isomorphism $\pi_2(\vec{x}, \vec{x}) \cong H_2(Y, \partial Y)$.*

Proof Recall that $\pi_2(\vec{x}, \vec{x}) \cong H_2\left(\Sigma^\circ \times [0,1], (\vec{\alpha} \times \{1\}) \cup (\vec{\beta} \times \{0\})\right)$. From the long exact sequence for the pair $\left(\Sigma^\circ \times [0,1], (\vec{\alpha} \times \{1\}) \cup (\vec{\beta} \times \{0\})\right)$, since $H_2(\Sigma^\circ) =$

0, we have

$$0 \longrightarrow H_2\left(\Sigma^\circ \times [0,1], (\vec{\alpha} \times \{1\}) \cup (\vec{\beta} \times \{0\})\right) \longrightarrow H_1\left(\vec{\alpha} \times \{1\} \cup (\vec{\beta} \times \{0\})\right) \longrightarrow H_1(\Sigma^\circ)$$

$$H_2\left(\Sigma^\circ \times [0,1], (\vec{\alpha} \times \{1\}) \cup (\vec{\beta} \times \{0\})\right) \longrightarrow \ker\left(\begin{pmatrix} H_1(\vec{\alpha}) \\ \oplus \\ H_1(\vec{\beta}) \end{pmatrix} \to H_1(\Sigma^\circ)\right) \xrightarrow{\ 0\ } H_1(\Sigma^\circ)$$

$$\ker\left(\begin{pmatrix} H_1(\vec{\alpha}) \\ \oplus \\ H_1(\vec{\beta}) \end{pmatrix} \to H_1(\Sigma^\circ)\right) = H_2(Y, \partial Y)$$

Since $H_2(Y, \partial Y) = \ker\left(H_1(\vec{\alpha}) \oplus H_1(\vec{\beta}) \to H_1(\Sigma/\partial\Sigma)\right)$ (Section 2.2), the result follows. □

Notice that, as a consequence of the proof, a homology class is completely determined by its domain. We shall use the words interchangeably.

We next turn to the question of when $\pi_2(\vec{x}, \vec{y})$ is empty. The situation is similar to the closed case as described in [OS04d, Section 2] or [Lip, Section 2] Given intersection points $\vec{x} = \{x_i\}$ and $\vec{y} = \{y_i\}$, choose g arcs γ_α in the α-circles connecting the x_i to the y_i. Choose g arcs γ_β in $\vec{\beta} \cup (\partial\Sigma \backslash \mathfrak{z})$ connecting the x_i to the y_i. Then, $\gamma_\alpha - \gamma_\beta$ defines a one-chain in Σ°. Let $\varepsilon(\vec{x}, \vec{y})$ denote the image of $\gamma_\alpha - \gamma_\beta$ in $\operatorname{coker}\left(H_1(\vec{\alpha}) \oplus H_1(\vec{\beta}) \to H_1(\Sigma^\circ)\right) = H_1(Y, \partial Y)$. We will see that ε is the obstruction to the existence of (topological) curves connecting $\vec{x}$ to $\vec{y}$.

Lemma 2.6.2 *For a pair of intersection points $\vec{x}$ and $\vec{y}$, $\varepsilon(\vec{x}, \vec{y}) = 0$ if and only if $\pi_2(\vec{x}, \vec{y})$ is nonempty.*

Proof If $\pi_2(\vec{x}, \vec{y})$ is nonempty then let $A \in \pi_2(\vec{x}, \vec{y})$. Then ∂A is a chain defining $\varepsilon(\vec{x}, \vec{y})$, so $\varepsilon(\vec{x}, \vec{y})$ is zero in homology. Conversely, if $\varepsilon(\vec{x}, \vec{y}) = 0$ then for an appropriate choice of γ_α and γ_β, $\gamma_\alpha - \gamma_\beta$ bounds in Σ. It is easy to see that the domain bounded by $\gamma_\alpha - \gamma_\beta$ corresponds to an element of $\pi_2(\vec{x}, \vec{y})$. □

Next, we construct a map from intersection points $\vec{x}$ to $\text{Spin}^{\mathbb{C}}$-structures $s_{\mathfrak{z}}(\vec{x})$ on Y. We will see that $\vec{x}$ and $\vec{y}$ represent the same $\text{Spin}^{\mathbb{C}}$-structure if and only if $\varepsilon(\vec{x}, \vec{y}) = 0$. In fact, $\text{PD}(\varepsilon(\vec{x}, \vec{y})) = s_{\mathfrak{z}}(\vec{x}) - s_{\mathfrak{z}}(\vec{y}) \in H^2(Y)$. (Here, PD denotes the

Poincaré duality isomorphism $H_1(Y, \partial Y) \to H^2(Y)$.) The construction is a simple adaptation from [OS04d, Section 2].

Recall from [Tur97] that a non-vanishing vector field v on Y specifies a $\text{Spin}^{\mathbb{C}}$-structure as follows. The choice of v, together with the orientation of Y and a Riemannian metric on Y reduces the structure group of TY to $U(1)$. The inclusion $U(1) \hookrightarrow \text{SO}(3)$ lifts to the standard inclusion $U(1) \hookrightarrow U(2) = \text{Spin}^{\mathbb{C}}(3)$. This gives a $\text{Spin}^{\mathbb{C}}$-structure on TY. It is not hard to check that two vector fields specify the same $\text{Spin}^{\mathbb{C}}$-structure if and only if they are homotopic through nonvanishing vector fields in the complement of some ball (or equivalently, finite disjoint union of balls). So, it remains for us to construct a non-vanishing vector field on Y, well-defined in the complement of some balls.

For convenience, fix a Morse function f on Y inducing the Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta})$ as in Section 2.2. Fix an intersection point $\vec{x} = \{x_i\}_{i=1}^{g}$. For notational convenience, assume that $x_i \in \alpha_i \cap \beta_{i+k}$ (so the first k β-arcs are not used in $\vec{x}$.). The basepoint $\mathfrak{z}$ lies on a flow from the index 0 critical point of f to the index 3 critical point; let B_0 denote a tubular neighborhood of this flow line. For $i = 1, \cdots, g$, the point x_i lies on a flow line from the i^{th} index 1 critical point of f to the $(i+k)^{\text{th}}$ index 2 critical point of f; let B_i be a small tubular neighborhood of this flow line. Finally, for $i = 1, \cdots, k$ let B_i denote a hemiball neighborhood of the i^{th} index 2 critical point of f (which lies on ∂Y). Let $B = \bigcup_{i=1}^{g+k/2} B_i$. The gradient ∇f of f is non-vanishing on $Y \backslash B$. Further, $\nabla f|_{Y \backslash B}$ admits an extension to all of Y as a non-vanishing vector field, and so specifies a $\text{Spin}^{\mathbb{C}}$-structure $s_{\mathfrak{z}}(\vec{x})$. It is clear that $s_{\mathfrak{z}}(\vec{x})$ does not depend on the choice of Morse function f inducing $(\Sigma, \vec{\alpha}, \vec{\beta})$. We may sometimes write $\vec{x} \in \mathfrak{s}$ to mean $s_{\mathfrak{z}}(\vec{x}) = \mathfrak{s}$.

The proof of the following lemma is the same as [OS04d, Lemma 2.19] or [Lip, Lemma 2.2], and we refer the interested reader there.

Lemma 2.6.3 *For intersection points $\vec{x}$ and $\vec{y}$, $\varepsilon(\vec{x}, \vec{y}) = 0$ if and only if $s_{\mathfrak{z}}(\vec{x}) = s_{\mathfrak{z}}(\vec{y})$.*

Next we turn to an issue which does not exist in Heegaard-Floer for closed 3-manifolds: how domains interact with $\partial\Sigma$. The role of this discussion in the present paper will be minimal, but these results are almost certain to be useful in future developments. Call an element A of $\pi_2(\vec{x},\vec{y})$ *provincial* if in ∂A the multiplicity of any arc in $\partial\Sigma$ is zero or, equivalently, if the coefficient in A of any component of $\Sigma\setminus(\vec{\alpha}\cup\vec{\beta})$ adjacent to $\partial\Sigma$ is zero. Let $\pi_2^\partial(\vec{x},\vec{y})$ denote the subset of $\pi_2(\vec{x},\vec{y})$ consisting of provincial homology classes. Notice that, as with $\pi_2(\vec{x},\vec{y})$, if $\pi_2^\partial(\vec{x},\vec{y})$ is nonempty then $\pi_2^\partial(\vec{x},\vec{y})\cong\pi_2^\partial(\vec{x},\vec{x})$, non-canonically.

Lemma 2.6.4 $\pi_2^\partial(\vec{x},\vec{x})\cong H_2(Y)$.

Proof Observe that $\pi_2^\partial(\vec{x},\vec{x})=H_2(\Sigma\times[0,1],\vec{\alpha}\times\{1\}\cup\vec{\beta}\times\{0\})$. The proof is then the same as the proof of Lemma 2.6.1, replacing Σ° with Σ everywhere. $\square$

Next we discuss when $\pi_2^\partial(\vec{x},\vec{y})$ is nonempty. Let $B(\vec{x})$ denote the set of β-arcs containing $\vec{x}$ (so $B(\vec{x})$ is a k element subset of $\{\beta_1,\cdots,\beta_{2k}\}$). Observe that if $\pi_2^\partial(\vec{x},\vec{y})\neq\emptyset$ then certainly $B(\vec{x})=B(\vec{y})$. Suppose $B(\vec{x})=B(\vec{y})$. Then we can define an obstruction $\varepsilon^\partial(\vec{x},\vec{y})$ to the existence of elements in $\pi_2^\partial(\vec{x},\vec{y})$ as follows. Let $\{a_i\}$ be a g-tuple of arcs in $\vec{\alpha}$ with $\partial(a_1+\cdots+a_g)=\vec{y}-\vec{x}$. Let $\{b_i\}$ be a g-tuple of arcs in $\vec{\beta}$ with $\partial(b_1+\cdots+b_g)=\vec{y}-\vec{x}$. Then $(a_1+\cdots+a_g)-(b_1+\cdots b_g)$ is a cycle in Σ. Let $\varepsilon^\partial(\vec{x},\vec{y})$ denote the image of this cycle in $H_1(Y)=H_1(\Sigma)/\left(H_1(\vec{\alpha})+H_1(\vec{\beta})\right)$.

Fix a Morse function f and metric g on Y inducing $(\Sigma,\vec{\alpha},\vec{\beta})$. An equivalent description of ε^∂ is given by considering the g flow lines $F(\vec{x})$ of f containing $\vec{x}$, and the g flow lines $F(\vec{y})$ of f containing $\vec{y}$. Then $\varepsilon^\partial(\vec{x},\vec{y})=F(\vec{y})-F(\vec{x})$.

Fix a $\text{Spin}^{\mathbb{C}}$-structure $\mathfrak{t}$ on a collar neighborhood D of ∂Y, such that $\mathfrak{t}$ extends over Y. (The $\text{Spin}^{\mathbb{C}}$-structure $\mathfrak{t}$ extends over Y if and only if $\langle c_1(\mathfrak{t}),[\partial Y]\rangle=0$. Since ∂Y is connected, this characterizes $\mathfrak{t}|_{\partial Y}$.) Let $\text{Spin}^{\mathbb{C}}(Y,\partial Y)$ denote the set of distinct extensions of $\mathfrak{t}$ over Y. Using the fact that $\text{Spin}^{\mathbb{C}}(n)\to\text{SO}(n)$ is a fibration with fiber S^1, it is not hard to see that $\text{Spin}^{\mathbb{C}}(Y,\partial Y)$ is an affine copy of $H^2(Y,\partial Y)\cong H_1(Y)$.

Fix a k-tuple of β-arcs B; B corresponds also to a k-tuple of index 1 critical points of $f|_{\partial Y}$. Let $\text{crit}(B)$ denote the union of this k-tuple with the index 0 and 2 critical points of $f|_{\partial Y}$. The $\text{Spin}^{\mathbb{C}}$-structure $\mathfrak{t}$ on D is specified by a non-vanishing vector field on D; we can choose a vector field which agrees with ∇f away from $\text{crit}(B)\times[0,1]\subset D=\partial Y\times[0,1]$. Then, given an intersection point $\vec{x}$ with $B(\vec{x})=B$, the $\text{Spin}^{\mathbb{C}}$-structure specified by the restriction of ∇f to the complement of the flow lines containing $\vec{x} \cup \{\mathfrak{z}\}$ extends $\mathfrak{t}$. Denote this extension by $s^{\partial}_{\mathfrak{z}}(\vec{x})$. We have the following lemma, the proof of which we leave to the reader.

Lemma 2.6.5 *For intersection points $\vec{x}$ and $\vec{y}$ with $B(\vec{x}) = B(\vec{y})$, the following conditions are equivalent.*

- $\varepsilon^{\partial}(\vec{x}, \vec{y}) = 0$.
- $s^{\partial}_{\mathfrak{z}}(\vec{x}) = s^{\partial}_{\mathfrak{z}}(\vec{y})$.
- $\pi^{\partial}_{2}(\vec{x}, \vec{y}) \neq \emptyset$.

Further, $\left(s^{\partial}_{\mathfrak{z}}(\vec{x}) - s^{\partial}_{\mathfrak{z}}(\vec{y})\right) = \mathrm{PD}\left(\varepsilon^{\partial}(\vec{x}, \vec{y})\right) \in H^2(Y, \partial Y)$. *(Here,* $\mathrm{PD} : H_1(Y) \to H^2(Y, \partial Y)$ *denotes Poincaré duality.)*

2.7 Admissibility for Heegaard diagrams with boundary

As is standard for Heegaard-Floer homology, in order to insure that various counts of curves are finite we will have to impose certain "admissibility conditions" on the Heegaard diagrams under consideration. A satisfactory condition is the "weak admissibility" of [OS04d]. Since our setting is somewhat different from theirs, however, we will explain the requirements from the beginning, making the necessary minor adaptations to their proofs. (When proofs are exactly the same, we will simply refer to the original.)

Recall that homology classes in $\pi_2(\vec{x}, \vec{y})$ correspond to certain cellular chains in Σ. Also, unlike in [OS04d] or [Lip], $\pi_2(\vec{x}, \vec{y})$ denotes only those chains with local multiplicity 0 at $\mathfrak{z}$. The following definition is equivalent to the "weak admissibility for all $\mathrm{Spin}^{\mathbb{C}}$-structures" of [OS04d, Definition 4.10]:

Definition 2.7.1 *A pointed Heegaard diagram with boundary $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ is called weakly admissible if every element of $\pi_2(\vec{x}, \vec{x})$ has both positive and negative coefficients.*

The proof of the following proposition is exactly the same as [OS04d, Lemma 4.12].

Proposition 2.7.2 *A pointed Heegaard diagram with boundary* $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ *is weakly admissible if and only if there is an area form on* Σ *with respect to which every element of* $\pi_2(\vec{x}, \vec{x})$ *has zero signed area.*

Corollary 2.7.3 *Given intersection points* $\vec{x}$ *and* $\vec{y}$ *in a weakly admissible Heegaard diagram there are at most finitely many domains in* $\pi_2(\vec{x}, \vec{y})$ *all of whose coefficients are positive.*

Proof Fix an area form on Σ as in Proposition 2.7.2. Suppose $A, A' \in \pi_2(\vec{x}, \vec{y})$. Then $A' = A + B$ for some $B \in \pi_2(\vec{x}, \vec{x})$. It follows that A and A' have the same area. Obviously only finitely many domains with positive coefficients can have the same area. □

Next, we turn to existence and "uniqueness" of weakly admissible Heegaard diagrams.

Proposition 2.7.4 *Given any pointed Heegaard diagram with boundary* $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ *there is an isotopic weakly admissible pointed Heegaard diagram with boundary* $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$. *More precisely, the* $\vec{\alpha}'$ *and* $\vec{\beta}'$ *are isotopic to the* $\vec{\alpha}$ *and* $\vec{\beta}$ *in the complement of* $\mathfrak{z}$*, via an isotopy fixed near* $\partial\Sigma$.

Proof (Compare [OS04d, Lemma 5.4].) The idea of the proof is that admissibility can be ensured by deforming the α-curves so that $\mathfrak{z}$ lies on both sides of each α-curve.

Let D denote the component of $\Sigma \setminus (\vec{\alpha} \cup \vec{\beta})$ in which $\mathfrak{z}$ lies. Then ∂D is a linear combination of arcs in various α- and β-curves. Suppose that for each i there is an arc of α_i occurring with a negative coefficient in ∂D and another arc of α_i occurring with a positive coefficient in ∂D. (In this case, we will say that $\mathfrak{z}$ *lies on both sides of* α_i.)

Claim. If $\mathfrak{z}$ lies on both sides of each α_i then the Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ is weakly admissible.

To see this, let $P \in \pi_2(\vec{x}, \vec{x})$. Then ∂P is a linear combination of α- and β-curves. Since the β_i are linearly independent, if $P \neq 0$ then some α_i occurs with a non-zero coefficient in ∂P. Since the local multiplicity of P at $\mathfrak{z}$ is zero, it follows from the fact that $\mathfrak{z}$ lies on both sides of α_i that P has both positive and negative coefficients. This proves the claim.

It is easy to arrange that $\mathfrak{z}$ lies on both sides of each α_i, proving the proposition. $\square$

Proposition 2.7.5 *Suppose that* $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ *and* $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$ *are isotopic weakly admissible Heegaard diagrams. Then* $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ *and* $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$ *are isotopic, in the complement of* $\mathfrak{z}$*, through weakly admissible Heegaard diagrams.*

Proof (Compare [OS04d, Lemma 5.6 and Lemma 5.8].) Preliminary to the proof proper, notice that an element P of $\pi_2(\vec{x}, \vec{x})$ is completely determined by ∂P, a linear combination of α- and β-curves. It follows that there is a natural correspondence between elements of $\pi_2(\vec{x}, \vec{x})$ and $\pi_2(\vec{y}, \vec{y})$ for different intersection points $\vec{x}$ and $\vec{y}$, and a natural correspondence between elements of $\pi_2(\vec{x}, \vec{x})$ and $\pi_2(\vec{x}', \vec{x}')$ for intersection points in isotopic Heegaard diagrams.

Throughout the proof, objects without primes ($'$s) will correspond to the Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ and objects with primes to the Heegaard diagram $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$.

To make the proof more transparent, we will prove the following slightly weaker statement: given $P \in \pi_2(\vec{x}, \vec{x})$ we can find an isotopy between $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ and $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$ so that the image P'' of P in any intermediate Heegaard diagram has both positive and negative coefficients. (To prove the proposition, one needs to find a single such isotopy for all P simultaneously. How our argument generalizes to prove this will be clear.)

Fix a point w_+ (respectively w_-) in Σ at with P has positive (respectively negative) local multiplicity. Fix a point w'_+ (respectively w'_-) in Σ at with P' has positive (respectively negative) local multiplicity. For appropriate choice of $w'_\pm$ and $(\Sigma, \vec{\alpha}'', \vec{\beta}'', \mathfrak{z})$ it is possible to find an isotopy I_1 from $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ to $(\Sigma, \vec{\alpha}'', \vec{\beta}'', \mathfrak{z})$

and an isotopy I_2 from $(\Sigma, \vec{\alpha}'', \vec{\beta}'', \mathfrak{z})$ to $(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z})$ so that I_1 is supported in the complement of $w_\pm$ and I_2 is supported in the complement of $w'_\pm$.

Suppose that Q is the image of P in some Heegaard diagram occurring during I_2. It is clear that $n_{w_\pm}(Q) = n_{w_\pm}(P)$ so Q has both positive and negative coefficients. Suppose that Q' is the image of P in some Heegaard diagram occurring during I_2. It is clear that $n_{w'_\pm}(Q') = n_{w'_\pm}(P)$ so Q' has both positive and negative coefficients. This proves the claim. □

Chapter 3

Invariants of 2-manifolds

3.1 Ordered lists of Reeb chords

By an *ordered list of Reeb chords at east infinity* we mean a sequence of Reeb chords at east infinity separated by $<$ signs. If o denotes an ordered list of Reeb chords we define $|o|$ to be the number of Reeb chords appearing in o.

Fix an ordered list of Reeb chords $o = \gamma_{i_1} < \cdots < \gamma_{i_j}$, and fix also k distinct β-arcs at $-\infty$, $B = \{\beta_{j_1}, \cdots, \beta_{j_k}\}$. (Later, B will be the β-arcs appearing in some intersection point $\vec{x}$.)

We try to use o to obtain a sequence of $j+1$ k-tuples of β-arcs $B_0, \cdots, B_j$, which can be thought of as the β-arcs in the image of u at various heights. (See Figure 3.1.) If we are successful we will call the pair $\Gamma = (o, B)$ *admissible*; otherwise, Γ is *inadmissible*. Set $B_0 = B$. Inductively, if $\beta_{down}(\gamma_{i_\ell}) \in B_{\ell-1}$ and $\beta_{up}(\gamma_{i_\ell}) \notin B_{\ell-1} \setminus \{\beta_{\text{down}}(\gamma_{i_\ell})\}$ then B_ℓ is obtained from $B_{\ell-1}$ by replacing $\beta_{down}(\gamma_{i_\ell})$ with $\beta_{up}(\gamma_{i_\ell})$. If at any stage either $\beta_{down}(\gamma_{i_\ell}) \notin B_{\ell-1}$ or $\beta_{up}(\gamma_{i_\ell}) \in B_{\ell-1} \setminus \{\beta_{\text{down}}(\gamma_{i_\ell})\}$ then $\Gamma = (o, B)$ is inadmissible.

Remark. There are (many) ordered lists of Reeb chords o for which there does not exist B such that (o, B) is admissible.

Remark. Later, we will consider holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$, where Σ is a Heegaard diagram with boundary. Such curves will be asymptotic to various Reeb chords, at various heights (in $\mathbb{R}$). Ordered lists of Reeb chords will keep track

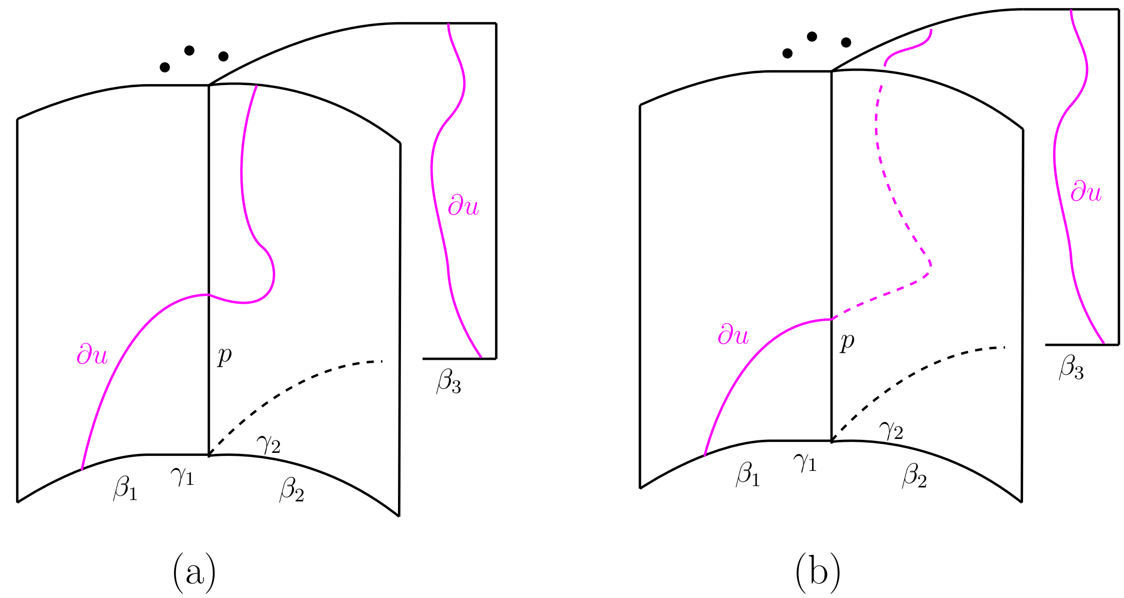


Figure 3.1: The pair $(\gamma_1, \{\beta_1, \beta_3\})$ shown in (a) is admissible. $B_0 = \{\beta_1, \beta_3\}$ and $B_1 = \{\beta_2, \beta_3\}$. The pair $(\gamma_{12}, \{\beta_1, \beta_3\})$ shown in (b) is inadmissible. $B_0 = \{\beta_1, \beta_3\}$ but $B_1 = \{\beta_3, \beta_3\}$, which is not allowed. Note that the pair $(\gamma_1 < \gamma_2, \{\beta_1, \beta_3\})$ is also inadmissible. (The vertical line in the middles of the pictures, labeled p, denotes east ∞.)

of the order induced by these heights. The motivation, then, for the concept of admissibility is as follows. Suppose $(\Sigma, \vec{\alpha}, \vec{\beta})$ is a closed manifold. In the cylindrical definition of Heegaard-Floer homology ([Lip]), the holomorphic maps

$$u : (S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta} \times \{0\} \times \mathbb{R}))$$

have the following property: for each $t \in \mathbb{R}$ and $1 \leq i \leq g$, $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of a single point. In other words, ∂u runs monotonically up each β-cylinder. In the relative case, we must loosen this condition to the condition that for each $1 \leq i \leq g + k$ and each $t \in \mathbb{R}$, $u^{-1}(\beta_i \times \{0\} \times \{t\})$ consists of at most a single point. Admissibility is a combinatorial requirement on the asymptotics for such a curve to be conceivable. The B_i can be thought of as the sequence of sets of β-arcs for which $u^{-1}(\beta_i \times \{0\} \times \{t\})$ is nonempty (as t goes from $-\infty$ to ∞). (See Figure 3.1 and Lemma 4.3.1.)

By a *two-level ordered list* of Reeb chords we mean a sequence of Reeb chords separated by $<$ and $<_\epsilon$ signs. From a two-level ordered list O we can obtain an ordered list $o(O)$ be replacing the $<_\epsilon$ signs with $<$ signs. In a two-level ordered list O, we refer to the maximal chains of Reeb chords separated by $<_\varepsilon$ signs as the *microscopic partitions* of O and the ordering of each microscopic partition the *microscopic ordering*. We call the partial order on O induced by the $<$ signs the *macroscopic ordering*.

Remark. When we consider holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$, part of the boundary of the spaces of curves will correspond to the heights (in $\mathbb{R}$) of different Reeb chords coming together. The $<_\epsilon$'s will be used to keep track of this boundary.

If O is a two-level ordered list of Reeb chords and B a k-tuple of β-arcs, we say (O, B) is admissible if $(o(O), B)$ is admissible.

If (O, B) is admissible, B_i is defined to be the i^{th} k-tuple of β-arcs induced by $(o(O), B)$. Define $|O| = |o(O)|$. We say that O *collapses to* O' if O' can be obtained from O by replacing some $<$ signs with $<_\varepsilon$ signs. Note that if O collapses to O' then (O, B) is admissible if and only if (O', B) is.

We will use the symbol $\prec$ to denote either $<$ or $<_\epsilon$, so $\gamma_{i_1} \prec \cdots \prec \gamma_{i_m}$ is the general two-level order with m terms.

Given admissible $\Gamma_1 = (O^1, B^1)$ and $\Gamma_2 = (O^2, B^2)$, if $B^1_{|O^1|} = B^2$ define $\Gamma_1 < \Gamma_2$ to be the admissible pair $(O_1 < O_2, B^1)$, where $O_1 < O_2$ is the ordered list of Reeb chords obtained by concatenating O_1 and O_2 with a $<$ sign in between. Let $\mathcal{G}$ denote the collection of all admissible pairs. We make $\mathcal{A} = \mathbb{F}_2[\mathcal{G}]$ into a ring by defining $[\Gamma_1] \cdot [\Gamma_2] = [\Gamma_1 < \Gamma_2]$ if $\Gamma_1 < \Gamma_2$ is defined and 0 otherwise.

Remark. The unit in $\mathcal{A}$ is $\sum_B (\emptyset, B)$.

There is an operation join : $\mathcal{A} \to \mathcal{A}$ given by

$$\operatorname{join}(O, B) = \sum_j (\gamma_{i_1} \prec \cdots \prec \gamma_{i_{j-1}} \prec \gamma_{i_{j+1}} \uplus \gamma_{i_j} \prec \gamma_{i_{j+1}} \prec \cdots \prec \gamma_m, B)$$

where $O = (\gamma_{i_1} \prec \cdots \prec \gamma_m)$ and the sum is over those j such that $\gamma_{i_j} <_\varepsilon \gamma_{i_{j+1}}$ in O, $\gamma_{i_{j+1}} \uplus \gamma_{i_j}$ is defined, and the pair on the right hand side is admissible.

Remark. Later (Proposition 4.6.1) we will see that part of the boundary of the space of curves in $\Sigma\times[0,1]\times\mathbb{R}$ corresponds to a Reeb chord splitting into two Reeb chords. The operation join will be used to record this part of the boundary.

Define a second operation decol : $\mathcal{A}\to\mathcal{A}$ by

$$\mathrm{decol}(O,B)=\sum_{\gamma_{i_j}<_\varepsilon\gamma_{i_{j+1}}\text{occurs in}O}(\gamma_{i_1}\prec\cdots\prec\gamma_{i_{j-1}}\prec\gamma_{i_j}<\gamma_{i_{j+1}}\prec\gamma_{i_{j+1}}\prec\cdots\prec\gamma_m,B).$$

That is, $\mathrm{decol}(O,B)$ is the sum of all possible ways of replacing a single $<_\varepsilon$ in O with a $<$ sign. (The symbol decol stands for "decollapse.")

Remark. Later (Proposition 4.6.1) we will see that part of the boundary of the space of curves in $\Sigma\times[0,1]\times\mathbb{R}$ corresponds to the heights of two Reeb chords coming together. The operation decol will be used to record this part of the boundary.

Note that we can regard any element of $\mathcal{A}$ as a subset of $\mathcal{G}$. So, the expressions $(O,B)\in\mathrm{decol}(O',B')$ and $(O,B)\in\mathrm{join}(O',B')$ have a natural meaning. (In fact, since the terms in the sum defining $\mathrm{decol}(O',B)$ (respectively $\mathrm{join}(O',B)$) are distinct, $(O,B)\in\mathrm{decol}(O',B)$ (respectively $(O,B)\in\mathrm{join}(O',B)$) if (O,B) appears in the sum defining $\mathrm{decol}(O,B)$ (respectively $\mathrm{join}(O',B)$).) Observe that $\mathrm{decol}(O,B)\cap\mathrm{join}(O,B)=\emptyset$.

Example. Consider the Heegaard diagram for $\mathbb{T}^2$ shown in Figure 3.1. There are Reeb chords $\gamma_1,\gamma_2,\gamma_3,\gamma_4=\gamma_1\uplus\gamma_2,\gamma_5=\gamma_2\uplus\gamma_3$ and $\gamma_6=\gamma_1\uplus\gamma_2\uplus\gamma_3$. The pair $(O,B)=(\gamma_2<\gamma_3<_\varepsilon\gamma_2<\gamma_4<_\varepsilon\gamma_3<\gamma_5,\{\beta_2\})$ is admissible. We have

$$\begin{aligned}\mathrm{decol}(O,B) &= (\gamma_2<\gamma_3<\gamma_2<\gamma_4<_\varepsilon\gamma_3<\gamma_5,\{\beta_2\})\\ &\quad+(\gamma_2<\gamma_3<_\varepsilon\gamma_2<\gamma_4<\gamma_3<\gamma_5,\{\beta_2\})\\ \mathrm{join}(O,B) &= 0.\end{aligned}$$

Remark. It is not hard to see that in the case $k=1$ (i.e., the surface is a torus), join is always zero. This is presumably related to the fact that, in the $k=1$ case, the boundary conditions correspond to a pair of nonsingular tori in $\mathrm{Sym}^g(\overline{\Sigma})$ (see Section 8.2). (Here, $\overline{\Sigma}$ denotes Σ with the puncture filled-in.) For $k>1$, the boundary conditions correspond to $2k$ choose k singular tori in $\mathrm{Sym}^g(\overline{\Sigma})$.

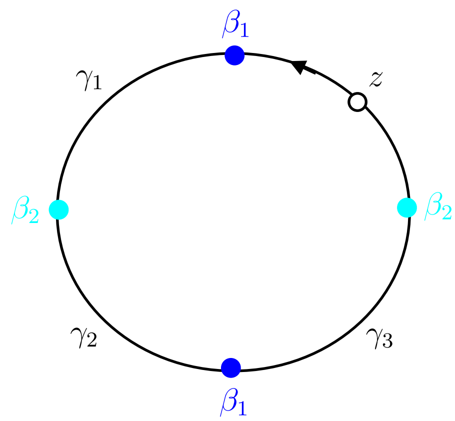


Figure 3.2: A pointed Heegaard diagram for $\mathbb{T}^2$.

3.2 The 2DHF DGA

Define a differential on $\mathcal{A}$ as follows: given an admissible pair Γ, define

$$d(\Gamma) = \text{join}(\Gamma) + \text{decol}(\Gamma).$$

Proposition 3.2.1 *d^2=0.*

Proof It is clear, in fact, that join and decol form a pair of commuting differentials, i.e.,

$$\begin{aligned} \text{join}(\text{join}(\Gamma)) &= 0 \\ \text{decol}(\text{decol}(\Gamma)) &= 0 \\ \text{join}(\text{decol}(\Gamma)) + \text{decol}(\text{join}(\Gamma)) &= 0. \end{aligned}$$

□

Proposition 3.2.2 *d satisfies the Leibniz rule. That is, $d(\Gamma_1\Gamma_2) = d(\Gamma_1)\Gamma_2 + \Gamma_1 d(\Gamma_2)$.*

Proof It is clear from their definitions that join and decol each satisfy the Leibniz rule. □

Remark. M. Hedden and E. Eftekhary point out that there is a homological grading gr on $\mathcal{A}$, defined by $\mathrm{gr}(O, B)$ is the number of $<_\varepsilon$'s appearing in O.

3.3 The homology of $\mathcal{A}$

Observe that to every two-level ordered list of Reeb chords O one can associate a homology class $H(O) \in H_1(C \setminus \{\mathfrak{z}\}, \boldsymbol{b}) = \mathbb{Z}^{|\boldsymbol{b}|}$. In fact, since a Reeb chord can only appear a non-negative number of times, $H(O) \in \mathbb{N}^{|\boldsymbol{b}|}$.[1] Let $H(\Gamma = (O, B)) = H(O)$. The map H makes $\mathcal{A}$ into a $\mathbb{N}^{|\boldsymbol{b}|}$-graded algebra. The differential on $\mathcal{A}$ preserves this grading in the sense that $H(d(\Gamma)) = H(\Gamma)$. (That is, it is an "internal" rather than "homological" grading. Later, we will be interested not in graded $\mathcal{A}$-modules but rather in filtered $\mathcal{A}$-modules, so the reader should perhaps think of H as giving a $\mathbb{N}^{|\boldsymbol{b}|}$-filtration rather than a grading.) So, $H(\mathcal{A}) = \oplus_{\vec{n}} H(\mathcal{A}_{\vec{n}})$, where $\mathcal{A}_{\vec{n}}$ is the part of $\mathcal{A}$ in grading $\vec{n}$. Each $\mathcal{A}_{\vec{n}}$ further decomposes according to the B in $\Gamma = (O, B)$, as $\mathcal{A}_{\vec{n},B}$.

There is a filtration on $\mathcal{A}_{\vec{n},B}$ by the number of Reeb chords appearing in O, that is, the total length of O. Let $\mathrm{gr}(\mathcal{A}_{\vec{n},B})$ denote the associated graded complex. There is a spectral sequence from the homology of $\mathrm{gr}(\mathcal{A}_{\vec{n},B})$ converging to the homology of $\mathcal{A}_{\vec{n},B}$. The differential on $\mathrm{gr}(\mathcal{A}_{\vec{n},B})$ is just decol. The complex $\mathrm{gr}(\mathcal{A}_{\vec{n},B})$ is a direct sum of complexes of the form $\left(\mathbb{F}_2 \xrightarrow{\mathrm{id}} \mathbb{F}_2\right)^{\otimes \ell}$, where (O, B) lies in $\left(\mathbb{F}_2 \xrightarrow{\mathrm{id}} \mathbb{F}_2\right)^{\otimes |O|}$. Thus, the homology of $\mathrm{gr}(\mathcal{A}_{\vec{n},B})$ is a free $\mathbb{F}_2$-vector space on admissible pairs (O, B) where O is either empty or consists of a single Reeb chord, and $H(O) = \vec{n}$. For each pair $\vec{n}, B$ there is at most one such (O, B). So, there are no higher differentials in the spectral sequence, and the homology of $\mathcal{A}$ is the free $\mathbb{F}_2$-vector space on admissible pairs (O, B) where O consists of either a single Reeb chord or the empty list of Reeb chords. Notice, in particular, that the homology of $\mathcal{A}$ is finite-dimensional.

For computations of the homology of the three-dimensional invariant below, we would like to use spectral sequence arguments like the one we used above. For technical reasons, to do so we must work not with $\mathcal{A}$ but rather with the completion $\widehat{\mathcal{A}}$ of $\mathcal{A}$ with respect to the filtration H.

[1] Here and later $\mathbb{N} = \{0, 1, 2, \cdots\}$ denotes the set of non-negative integers.

Let $\mathcal{A}_0$ denote the part of $\mathcal{A}$ in H-filtration level zero. That is, $\mathcal{A}_0$ is generated by pairs $(\emptyset, B)$. Recall from the definition that

$$(\emptyset, B_1) \cdot (\emptyset, B_2) = \begin{cases} 0 & \text{if} B_1 \neq B_2 \\ (\emptyset, B_1) & \text{if} B_1 = B_2 \end{cases} .$$

Chapter 4

Structure of the moduli spaces

4.1 Almost complex structures, moduli spaces and transversality

Fix a neighborhood U_p of the puncture p in Σ. Let $j_{\mathbb{D}}$ denote the standard complex structure on $[0,1] \times \mathbb{R}$ and j_Σ some complex structure on Σ. We will work with almost complex structures J on $\Sigma \times [0,1] \times \mathbb{R}$ satisfying the following properties.

(**J1**) The projection map $\pi_{\mathbb{D}}$ is $(J, j_{\mathbb{D}})$-holomorphic.

(**J2**) For (s,t) coordinates on $[0,1] \times \mathbb{R}$, $J\frac{\partial}{\partial t} = -\frac{\partial}{\partial s}$.

(**J3**) $J|_{\pi_\Sigma^{-1}(U_p)}$ is split, i.e., $J|_{\pi_\Sigma^{-1}(U_p)} = j_\Sigma \times j_{\mathbb{D}}$.

(**J4**) J is $\mathbb{R}$-translation invariant.

The choice of such a J is equivalent to choosing a family $j_{\Sigma,s}$ of almost complex structures on Σ parameterized by $s \in [0,1]$ (with $j_{\Sigma,s}|_{U_p} = j_\Sigma$). Observe also that with respect to such a J, the fibers of π_Σ and $\pi_{\mathbb{D}}$ are J-holomorphic. Further, for ω_Σ and $\omega_{\mathbb{D}}$ positively-oriented area forms on Σ and $\mathbb{D}$ respectively, such a J is tamed by the split symplectic form $\pi_\Sigma^*\omega_\Sigma + \pi_{\mathbb{D}}^*\omega_{\mathbb{D}}$ on $\Sigma \times [0,1] \times \mathbb{R}$.

Let $\overline{\Sigma}$ denote Σ with the puncture filled-in, so $\overline{\Sigma}$ is a closed surface. Observe that any almost complex structure J satisfying (**J3**) extends over $\overline{\Sigma} \times [0,1] \times \mathbb{R}$; we will denote the extension by J as well.

We will be interested in J-holomorphic maps

$$u : (S, \partial S) \to \Big((\Sigma \setminus \mathfrak{z}) \times [0,1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta} \times \{0\} \times \mathbb{R})\Big)$$

which extend to proper, finite-energy (in the sense of [BEH$^+$03, Section 5.3]) J-holomorphic $u : \overline{S} \to \overline{\Sigma} \times [0,1] \times \mathbb{R}$ for some partial compactification $\overline{S}$ of S. (Here, S is a surface with boundary and punctures on the boundary.) It follows from [BEH$^+$03, Proposition 5.8] that at each puncture of S, u is asymptotic to either $x \times [0,1]$ at $+\infty$ or $-\infty$ for some $x \in \vec{\alpha} \cap \vec{\beta}$ or to $\gamma \times \{0\} \times \{t\}$ for some Reeb chord γ at east infinity and $t \in \mathbb{R}$. For a given almost complex structure J, let $\mathcal{M}(J)$ denote the moduli space of all such curves. The space $\mathcal{M}(J)$ has different components corresponding to different sources S, asymptotics, and homology classes of maps, but for the time being we do not care. Let $\mathcal{M}_{\mathbb{D}-\text{flat}}(J)$ denote the subspace of $\mathcal{M}(J)$ of those curves $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ for which there is some component of S on which $\pi_{\mathbb{D}}$ is constant. Let $\mathcal{M}_{\Sigma-\text{flat}}(J)$ denote the subspace of $\mathcal{M}(J)$ of those curves $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ for which there is some component of S which is not a twice-punctured disk on which $\pi_\Sigma \circ u$ is constant.

The following proposition is proved in exactly the same way as [Lip, Proposition 3.8].

Proposition 4.1.1 *For a generic choice of J, $\mathcal{M}(J) \setminus (\mathcal{M}_{\mathbb{D}-\text{flat}}(J) \cup \mathcal{M}_{\Sigma-\text{flat}}(J))$ is transversely cut out by the $\overline{\partial}$-equation, and as such is a smooth orbifold.*

We next turn to maps to $\partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R}$, which will appear as parts of the limits of sequences of maps to $\Sigma \times [0,1] \times \mathbb{R}$. We begin with some notation. Let $\partial\Sigma$ be a circle, which we identify with east ∞ in Σ. We will consider holomorphic curves in $(\partial\Sigma \setminus \mathfrak{z}) \times \mathbb{R} \times [0,1] \times \mathbb{R}$ with respect to the split complex structure. Let $\pi_\Sigma : (\partial\Sigma) \times \mathbb{R} \times [0,1] \times \mathbb{R} \to (\partial\Sigma) \times \mathbb{R}$ denote projection onto the first two factors and $\pi_{\mathbb{D}} : (\partial\Sigma) \times \mathbb{R} \times [0,1] \times \mathbb{R} \to [0,1] \times \mathbb{R}$ denote projection onto the last two factors; observe that both projections are holomorphic. We refer to $\partial\Sigma \times \mathbb{R} \times [0,1] \times \{+\infty\}$ as $+\infty$, $\partial\Sigma \times \mathbb{R} \times [0,1] \times \{-\infty\}$ as $-\infty$, $\partial\Sigma \times \{-\infty\} \times [0,1] \times \mathbb{R}$ as west ∞, and $\partial\Sigma \times \{+\infty\} \times [0,1] \times \mathbb{R}$ as east ∞. Recall that $\boldsymbol{b} = \vec{\beta} \cap \partial\Sigma$. Arcs in $\partial\Sigma$ between points in $\boldsymbol{b}$ specify *Reeb chords at (both) east and west ∞.*

The following lemma will be useful in restricting which types of degenerations can occur in codimension 1.

Lemma 4.1.2 *Let S denote a disk with at least three boundary punctures, and $u : (S, \partial S) \to (\partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R}, \boldsymbol{b} \times \mathbb{R} \times \{0\} \times \mathbb{R})$ a holomorphic map with respect to the split complex structure $j_\Sigma \times j_{\mathbb{D}}$ on $(S^1 \times \mathbb{R}) \times ([0,1] \times \mathbb{R})$ asymptotic to certain Reeb chords at east and west ∞. Then the linearization $D_u\overline{\partial}$ at u of the $\overline{\partial}$-operator is surjective.*

Proof For notational convenience, let $M_1 = \partial\Sigma \times \mathbb{R}$ and $M_2 = [0,1] \times \mathbb{R}$. Recall that

$$D_u\overline{\partial} : \Gamma_L \left(u^*T(M_1 \times M_2)\right) \to \Gamma\left(\Lambda^{0,1}u^*T(M_1 \times M_2)\right),$$

where Γ and Γ_L denote certain weighted Sobolev spaces of sections, which in the case of Γ_L are required to be tangent to the Lagrangian planes $\mathbf{b} \times \mathbb{R} \times \{0\} \times \mathbb{R}$. Observe that

$$u^*T(M_1 \times M_2) = ((\pi_\Sigma \circ u)^* \mathrm{TM}_1) \oplus ((\pi_{\mathbb{D}} \circ u)^* \mathrm{TM}_2),$$

as complex vector bundles with Lagrangian subbundles over ∂S. It suffices to prove, therefore, that both $D_{\pi_\Sigma \circ u}\overline{\partial} : \Gamma_L\left((\pi_\Sigma \circ u)^* \mathrm{TM}_1\right) \to \Gamma\left(\Lambda^{0,1}(\pi_\Sigma \circ u)^* \mathrm{TM}_1\right)$ and $D_{\pi_{\mathbb{D}} \circ u}\overline{\partial} : \Gamma_L\left((\pi_{\mathbb{D}} \circ u)^* \mathrm{TM}_1\right) \to \Gamma\left(\Lambda^{0,1}(\pi_{\mathbb{D}} \circ u)^* \mathrm{TM}_1\right)$ are surjective.

We check $D_{\pi_{\mathbb{D}} \circ u}\overline{\partial}$ is surjective first. Observe that since $\pi_{\mathbb{D}} \circ u$ is constant, $(\pi_{\mathbb{D}} \circ u)^* \mathrm{TM}_1$ is, in fact, a trivial line bundle. Let $\mathcal{O}_{\mathbb{P}^1}$ denote the trivial holomorphic line bundle over complex projective space $\mathbb{P}^1$. Viewing $\mathbb{D}$ as half of $\mathbb{P}^1$, it follows from a standard doubling argument (see for instance [HLS97, Section 4]) that the cokernel of $D_{\pi_{\mathbb{D}} \circ u}\overline{\partial}$ embeds in $\mathrm{coker}(\overline{\partial}) : \Gamma(\mathcal{O}_{\mathbb{P}^1}) \to \Gamma\left(\Lambda^{0,1}(\mathcal{O}_{\mathbb{P}^1})\right)$. By Hodge theory, this is the sheaf cohomology group $H^1(\mathcal{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1; \mathbb{C}) = 0$. This proves surjectivity of $D_{\pi_{\mathbb{D}} \circ u}\overline{\partial}$.

The proof of surjectivity of $D_{\pi_\Sigma \circ u}\overline{\partial}$ is similar. Again, $(\pi_\Sigma \circ u)^* \mathrm{TM}_2$ doubles to a holomorphic line bundle L over $\mathbb{P}^{\mathbb{1}}$, so that $\mathrm{coker}(D_{\pi_\Sigma \circ u}\overline{\partial})$ is a subspace of $\mathrm{coker}\left(\overline{\partial} : \Gamma(L) \to \Gamma\left(\Lambda^{0,1}(L)\right)\right)$. This cokernel is isomorphic to $H^1(L)$. So, it suffices to show that $H^1(L)$ vanishes, or equivalently that the degree of L is non-negative.

By the Riemann-Roch theorem, the degree of L is $\frac{1}{2}\mathrm{ind}\big(\overline{\partial}:\Gamma(L)\to\Gamma(\Lambda^{0,1}(L))\big) - 1$ which by the nature of the doubling procedure is just $\mathrm{ind}(D_{\pi_\Sigma\circ u}\overline{\partial})-1$. Thus, it suffices to show that for $\pi_\Sigma\circ u$ holomorphic, if S has at least three boundary punctures then $\mathrm{ind}(D_{\pi_\Sigma\circ u}\overline{\partial})\geq 1$. But it follows from any of the standard formulas for the index of $D\overline{\partial}$ that $\mathrm{ind}(D_{\pi_\Sigma\circ u}\overline{\partial})$ is two less than twice the number of branch points of $\pi_\Sigma\circ u$ (where branch points on ∂S count for 1/2). This proves the result. □

Note that the only holomorphic disks in $(\partial\Sigma\times\mathbb{R}\times[0,1]\times\mathbb{R},\boldsymbol{b}\times\mathbb{R}\times\{0\}\times\mathbb{R})$ with fewer than three boundary punctures are trivial strips, mapped homeomorphically onto a strip in $(\partial\Sigma\times\mathbb{R},\mathbf{b}\times\mathbb{R})$.

4.2 Compactness and gluing

In this section we discuss the most general objects into which a sequence of holomorphic curves in $\Sigma\times[0,1]\times\mathbb{R}$ can degenerate. In the next section, we will discuss restrictions on what degenerations can, in fact, occur in codimension 1.

Degenerations will include curves mapped to "east infinity", and before discussing them we need appropriate notation.

By a *one-story holomorphic comb in* $\Sigma\times[0,1]\times\mathbb{R}$ we mean

- a map $u:(S_0,\partial S_0)\to((\Sigma\backslash\{\mathfrak{z}\})\times[0,1]\times\mathbb{R},(\vec{\alpha}\times\{1\}\times\mathbb{R})\cup(\vec{\beta}\times\{0\}\times\mathbb{R}))$, defined up to translation in $\mathbb{R}$, and
- maps $v_i:(S_i,\partial S_i)\to((\partial\Sigma\backslash\{\mathfrak{z}\})\times\mathbb{R}\times[0,1]\times\mathbb{R},\boldsymbol{b}\times\mathbb{R}\times\{0\}\times\mathbb{R})$, $i=1,\cdots,k$, defined up to translation in both $\mathbb{R}$-factors[1]

such that the following technical conditions are satisfied.

- Each S_i is a surface with boundary and punctures on the boundary.
- The map u extends to a proper map $\overline{u}:\overline{S}_0\to\overline{\Sigma}\times[0,1]\times\mathbb{R}$ where $\overline{S}_0$ is a partial compactification of S_0, and $\overline{u}$ is finite-energy in the sense of [BEH+03, Section 5.3].

[1]In this section, k has nothing to do with the genus of ∂Y.

- For each i, $\pi_\Sigma \circ v_i$ and $\pi_{\mathbb{D}} \circ v_i$ is proper.
- For each i, $\pi_{\mathbb{D}} \circ v_i$ is contained in a compact subset of $[0,1]\times\mathbb{R}$.
- The maps u and v_i $(i=1,\cdots,k)$ are *stable.* That is, there are no infinitesimal automorphisms of u or v_i $(i=1,\cdots,k)$.

The technical conditions imply that u is asymptotic to certain Reeb chords at $\pm\infty$ and Reeb chords $\gamma_{0,j}\times\{(0,t_{0,j})\}$, $j=1,\cdots,n_0$, at east ∞, and that each v_i is asymptotic to Reeb chords $\gamma'_{i,j}\times\{(0,t'_{i,j})\}$, $j=1,\cdots,n'_i$ at west ∞ and $\gamma_{i,j}\times\{(0,t_{i,j})\}$, $j=1,\cdots,n_i$ at east ∞. We further require that

- for $i=0,\cdots,k-1$, $n_i=n'_{i+1}$ and
- for an appropriate ordering of the punctures on each S_i, $\gamma_{i,j}=\gamma'_{i+1,j}$ and $t_{i,j}=t'_{i+1,j}$.

By an *ℓ-story holomorphic comb in* $\Sigma\times[0,1]\times\mathbb{R}$ we mean ℓ one-story holomorphic combs $\{u_1,v_{1,i}\},\cdots,\{u_\ell,v_{\ell,i}\}$ such that the asymptotics of u_i at $+\infty$ agree with the asymptotics of u_{i+1} at $-\infty$ for $i=1,\cdots,\ell-1$. That is, if u_i is asymptotic to $x_1\times[0,1],\cdots,x_m\times[0,1]$ at $+\infty$ then u_{i+1} is asymptotic to $x_1\times[0,1],\cdots,x_m\times[0,1]$ at $-\infty$.

Remark. The surfaces S_i may be nodal. We say that the comb is smooth if all of the S_i are smooth.

The topology on the space of holomorphic combs is a straightforward generalization of the topology on the space of multi-story holomorphic buildings described in [BEH$^+$03]. That is, roughly, a sequence of maps $u^n: S^n\to\Sigma\times[0,1]\times\mathbb{R}$ converges to a holomorphic comb $\{u_i: S_{0,j}\to\Sigma\times[0,1]\times\mathbb{R}, v_{i,j}: S_{i,j}\to\partial\Sigma\times\mathbb{R}\times[0,1]\times\mathbb{R}\}$ if after adding N marked points (some N) to stabilize all of the components of all the $S_{i,j}$, one can add N marked points to each S^n so that the sources S^n converge, in the sense of Deligne-Mumford, with the marked points and punctures, to $\{S_{i,j}\}$. In the process, a collection Γ of arcs and circles in the S^n collapse. The maps u^n should converge uniformly on compact subsets of $S^n\backslash\Gamma$ to $\{u_i, v_{i,j}\}$. Write $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ where Γ_1 corresponds to arcs collapsing to punctures of the $S_{0,j}$ mapped to $\pm\infty$, Γ_2

to arcs collapsing to punctures of the $S_{i,j}$ mapped to Reeb chords at east/west ∞, and the Γ_3 to nodes of the $S_{i,j}$. For convenience, assume $\Gamma_3 = \emptyset$. Then, near the Γ_1, $\pi_{\Sigma\times[0,1]} \circ u^n$ should converge uniformly, and $\pi_{\mathbb{R}} \circ u^n$ should converge uniformly on compact sets (up to vertical translation of each u^n). Near the Γ_2-arcs, up to vertical translation of each u^n, $\pi_{\mathbb{D}} \circ u^n$ should converge uniformly on compact sets, and up to horizontal translation of each u^n (i.e., in the $\mathbb{R}$-coordinate of a collar neighborhood of $\partial\Sigma$), $\pi_\Sigma \circ u^n$ should converge uniformly on compact sets.

A precise definition of convergence is that, after adding a sufficient number of marked points to the S^n and $S_{i,j}$,

- the $\pi_{\mathbb{D}} \circ u^n$ converge in the sense of [BEH+03, Section 7.3] to $\{\pi_{\mathbb{D}} \circ u_i, \pi_{\mathbb{D}} \circ v_{i,j}\}$
- the $\pi_\Sigma \circ u^n|_{(\pi_\Sigma\circ u^n)^{-1}(\bar{U}_p)}$ converge in the sense of [BEH+03, Section 9.1] to $\{\pi_\Sigma \circ u_i|_{(\pi_\Sigma\circ u_i)^{-1}(\bar{U}_p)}, \pi_\Sigma \circ v_{i,j}\}$ and
- the $u^n|_{(\pi_\Sigma\circ u^n)^{-1}(\Sigma\setminus U_p)}$ converge in the sense of [BEH+03, Section 7.3] to $\{u_i|_{(\pi_\Sigma\circ u_i)^{-1}(\Sigma\setminus U_p)}\}$.

(Here, U_p is the neighborhood of the puncture of Σ fixed in chapter 4.1.)

The next proposition gives the compactness result we will need.

Proposition 4.2.1 *Let $\{u^n : S^n \to \Sigma\times[0,1]\times\mathbb{R}\}_{n=1}^\infty$ be a sequence of holomorphic curves in the same homology class. Then there is a subsequence $\{u^{n_i}\}$ converging to a holomorphic comb. More generally, any sequence of holomorphic combs in the same homology class has a convergent subsequence.*

Proof We will prove the result for a sequence of holomorphic curves; the proof of the result for sequences of combs is essentially the same, but notationally more cumbersome. Actually, the definition of convergence essentially spells out how to deduce this result from the compactness results of [BEH+03].

By [BEH+03, Theorem 10.1], we can find a subsequence of $\{u^n\}$ for which $\pi_{\mathbb{D}} \circ u^n$ converges to some multi-story holomorphic building.[2] (In the process, we add

[2] The compactness theorems [BEH+03, Theorems 10.1 and 10.2] are stated in the absolute case (i.e., for curves without boundary). See Section 10.3 of that paper for the generalizations to the relative case.

enough marked points to the sources S^n to stabilize each component of the limit curve.) From now on, let $\{u^n\}$ denote this subsequence. Call the limit $\{\pi_{\mathbb{D}} \circ u_{a,i} : S_{a,i} \to \mathbb{D}\}_{i=1}^{\ell}$; here, u_i denotes the i^{th} level of the limit.

The $\pi_\Sigma \circ u^n|_{(\pi_\Sigma \circ u^n)^{-1}(\bar{U}_p)}$ form a sequence of holomorphic curves with Lagrangian boundary conditions ∂U_p. By [BEH+03, Theorem 10.2], we can find a convergent subsequence. (Again, more marked points are added to the sources in the process.) Let $\{\pi_\Sigma \circ u_{b,i} : S_{b,i} \to \bar{U}_p \subset \Sigma, \pi_\Sigma \circ v_{b,i,j} : T_{b,i,j} \to \partial\Sigma \times \mathbb{R}\}$ denote the limit; the meaning of the indexing needs to be explained. Forgetting the new marked points and collapsing unstable components gives a map from the source of the limit curve $(\bigcup_i S_{b,i}) \cup (\bigcup_{i,j} T_{b,i,j})$ to $\bigcup_i S_{a,i}$. This *defines* the index i. The index j comes from the level structure in Σ; as the notation indicates, the $S_{b,i}$ are the components mapped to Σ and the $T_{b,i,j}$ the components mapped to east ∞.

Observe that $\pi_{\mathbb{D}} \circ u_{a,i}$ is naturally defined on $\{S_{b,i}, T_{b,i,j}\}$. Set $u_{b,i}|_{(\pi_\Sigma \circ u_{b,i})^{-1}(U_p)} = (\pi_{\mathbb{D}} \circ u_{a,i}, \pi_\Sigma \circ u_{b,i})$ and $v_{b,i,j} = (\pi_{\mathbb{D}} \circ u_{a,i}, \pi_\Sigma \circ v_{b,i,j})$ on $\{S_{b,i}, T_{b,i,j}\}$.

Turning to the rest of Σ, let $L_n = \{\pi_{\mathbb{D}} \circ u^n\left((\pi_\Sigma \circ u^n)^{-1}(\partial U_p)\right)\}$. By considering a slightly larger neighborhood $V_p \supset \bar{U}_p$ in Σ, one sees that $\{L_n\}$ forms a convergent sequence of smooth curves. Then, $u^n|_{(\pi_\Sigma \circ u)^{-1}(\Sigma \setminus U_p)}$ is a sequence of holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$ with Lagrangian boundary conditions $(\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta} \times \{0\} \times \mathbb{R}) \cup (\partial U_p \times L_n)$. The compactness theorem [BEH+03, Theorem 10.2] now applies to give a convergent subsequence of the $u^n|_{(\pi_\Sigma \circ u)^{-1}(\Sigma \setminus U_p)}$, with limit $u_{c,i}|_{(\pi_\Sigma \circ u_{c,i})^{-1}(\Sigma \setminus U_p)}$.

Observe that $u_{b,i}|_{(\pi_\Sigma \circ u_{b,i})^{-1}(U_p)}$ and $u_{c,i}|_{(\pi_\Sigma \circ u_{c,i})^{-1}(\Sigma \setminus U_p)}$ fit together to form a smooth curve $u_{c,i}$. (This follows, again, by considering the neighborhood $V_p \supset \overline{U}_p$.)

We are left, finally, with a sequence u^n so that $u^n|_{(\pi_\Sigma \circ u)^{-1}(\Sigma \setminus U_p)}$, $\pi_{\mathbb{D}} \circ u^n$, and $\pi_\Sigma \circ u^n|_{(\pi_\Sigma \circ u^n)^{-1}(\bar{U}_p)}$ all converge. This proves the result. □

We next turn to the main gluing result necessary for our theory. We need a little more notation before stating the result. For a holomorphic map $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ let $\mathcal{M}(u)$ denote the moduli space of holomorphic maps $S \to \Sigma \times [0,1] \times \mathbb{R}$ homotopic to u, modulo the $\mathbb{R}$-action by translation. Similarly, for $v : S \to (\partial\Sigma) \times \mathbb{R} \times [0,1] \times \mathbb{R}$ holomorphic, let $\mathcal{M}(v)$ denote the moduli space of holomorphic maps $S \to (\partial\Sigma) \times$

$\mathbb{R} \times [0,1] \times \mathbb{R}$ homotopic to v, modulo the $\mathbb{R}^2$-action by translation in both $\mathbb{R}$-factors. (We allow the asymptotics to vary continuously during the homotopy.) If u is asymptotic to Reeb chords $\gamma_1, \cdots, \gamma_\ell$ at east ∞ then there is an evaluation map $\mathrm{ev} : \mathcal{M}(u) \to \mathbb{R}^\ell/\mathbb{R}$. (Here, $\mathbb{R}$ acts on $\mathbb{R}^\ell$ by simultaneous translation.) Similarly, if v is asymptotic to Reeb chords $\gamma_1^w, \cdots, \gamma_{\ell_w}^w$ at west ∞ and $\gamma_1^e, \cdots, \gamma_{\ell_e}^e$ at east ∞ then there are evaluation maps $\mathrm{ev}_w : \mathcal{M}(v) \to \mathbb{R}^{\ell_w}/\mathbb{R}$ and $\mathrm{ev}_e : \mathcal{M}(v) \to \mathbb{R}^{\ell_e}/\mathbb{R}$.

The proof of the following proposition is a trivial adaptation of the gluing arguments of [Bou02, Section 5.3] to the relative case, or [Lip, Proposition A.1] to the Morse-Bott case; the local nature of the gluing result means that the fact that our spaces have "two infinities" does not introduce new difficulties. (There would be new difficulties if holomorphic curves could approach both infinities at once.)

Proposition 4.2.2 *Let $\{u_i, v_{i,j}\}$ ($i = 1, \cdots, m$, $j = 1, \cdots, n_i$) be a holomorphic comb. Suppose that the linearized $\overline{\partial}$-operator is surjective at each u_i and $v_{i,j}$. Suppose further that*

$$[d(\mathrm{ev})(T_{u_i}\mathcal{M}(u_i))] \pitchfork \left[d(\mathrm{ev}_w)(T_{v_{i,1}}\mathcal{M}(v_{i,1}))\right]$$

for each i, and

$$\left[d(\mathrm{ev}_e)(T_{v_{i,j}}\mathcal{M}(v_{i,j}))\right] \pitchfork \left[d(\mathrm{ev}_w)(T_{v_{i,j+1}}\mathcal{M}(v_{i,j+1}))\right]$$

for each i, j. Then near $\{u_i, v_{i,j}\}$ the moduli space of holomorphic curves is modeled on $\mathbb{R}^N \times [0,1)^M$ where $N = \sum_i \dim(T_{u_i}\mathcal{M}(u_i)) + \sum_{i,j} \dim(T_{v_{i,j}}\mathcal{M}(v_{i,j})) - \sum_{i,j}(k_{i,j} - 1)$ where $k_{i,j}$ is the number of Reeb chords of $v_{i,j}$ at west ∞, and $M = m + \sum_{i=1}^m n_i$.

In words, the proposition says that if all of the pieces of the holomorphic comb are transversely cut out by $\overline{\partial}$, and consecutive evaluation maps are transverse to each other then the moduli spaces can be glued.

Remark. During the invariance proof later we will be interested in holomorphic curves in $\Sigma \times [0,1] \times \mathbb{R}$ with a non-cylindrical almost complex structure or non-cylindrical boundary conditions, and in holomorphic curves in $\Sigma \times T$ for T a disk with more than two boundary punctures. There are obvious generalizations of the definition of a holomorphic comb and Proposition 4.2.1 to these settings. The obvious generalizations of Proposition 4.2.2 to these cases are also true — though we will actually need something slightly more general in chapter 7 when working with $\Sigma \times T$; the necessary modifications will be discussed when needed.

4.3 The Heegaard-Floer moduli spaces

By a (height 1) decorated source $S^{\diamond}$ we mean

- A Riemann surface S with boundary and punctures on the boundary.
- A labeling of each puncture of S with either $+\infty$, $-\infty$, or $e\infty$.
- A labeling of each puncture labeled $+\infty$ or $-\infty$ by a Reeb chord at $\pm\infty$ (i.e., a point $x \in \alpha_i \cap \beta_j$).
- A labeling of each puncture q labeled $e\infty$ by a Reeb chord $\gamma(q)$ at east ∞.
- A partition of each $\gamma(q)$ as $\gamma(q) = \gamma_1(q) \uplus \cdots \uplus \gamma_{k(q)}(q)$.

We will assume that exactly g punctures are labeled by $-\infty$ (respectively $+\infty$), and that the Reeb chords corresponding to $-\infty$ punctures (respectively $+\infty$ punctures) form an intersection point. Given intersection points $\vec{x}$ and $\vec{y}$, it then makes sense to talk about the decorated sources connecting $\vec{x}$ to $\vec{y}$.

Notice that for each q there is an induced two-level order on the $\gamma_i(q)$ by setting $\gamma_1(q) <_{\varepsilon} \gamma_2(q) <_{\varepsilon} \cdots <_{\varepsilon} \gamma_{k(q)}(q)$. Let $\gamma(S^{\diamond}) = \cup_q \{\gamma_1(q), \cdots, \gamma_{k(q)}(q)\}$

Given a decorated source $S^{\diamond}$, we say that a map $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ respects the decorations of $S^{\diamond}$ if the asymptotics of u are those specified by $S^{\diamond}$. That is, we assume that at each puncture labeled by $+\infty$ and y_j (respectively $-\infty$ and x_i) the map u is asymptotic to $y_j \times [0,1] \times \{+\infty\}$ (respectively $x_i \times [0,1] \times \{-\infty\}$), and at each puncture q labeled by $\gamma(q)$, u is asymptotic to $\gamma(q) \times \{0\} \times \{t_q\}$ for some $t_q \in \mathbb{R}$.

Note that every map $u : S^{\diamond} \to \Sigma \times [0,1] \times \mathbb{R}$ respecting the decorations belongs to some homology class in $\pi_2(\vec{x}, \vec{y})$, as defined in Section 2.6.

Given a map u respecting the decorations of $S^\diamond$ and a two-level order O of $\gamma(S^\diamond)$ we say that u is consistent with O if the macroscopic ordering of O is induced (from the standard ordering on $\mathbb{R}$) by $\pi_\mathbb{R} \circ u$ and the microscopic ordering of O is consistent with the microscopic orderings $\gamma_1(q) <_\varepsilon \gamma_2(q) <_\varepsilon \cdots <_\varepsilon \gamma_{k(q)}(q)$.

Fix a generic almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$ satisfying (**J1**)–(**J4**). Given a homology class $A \in \pi_2(\vec{x}, \vec{y})$ and a two-level order O let $\mathcal{M}^{A,O}$ denote the union over all decorated sources $S^\diamond$ of the moduli space of J-holomorphic curves $u : S^\diamond \to \Sigma \times [0,1] \times \mathbb{R}$, modulo vertical translation, respecting the decorations of $S^\diamond$, consistent with O, and satisfying the following technical conditions:

- The source S of u is smooth (not nodal).
- The map u is an embedding.
- The map u extends to a map $\overline{u} : \overline{S} \to \overline{\Sigma} \times [0,1] \times \mathbb{R}$ (for some partial compactification $\overline{S}$ of S) such that $\overline{u}$ has finite energy in the sense of [BEH+03, Section 5.3].
- There are no components of S on which $\pi_\mathbb{D} \circ u$ is constant.

Since we quotiented by the $\mathbb{R}$-action by translation, the expected dimension of $\mathcal{M}^{A,O}$ is $\text{ind}(A, O) - 1$.

Given an intersection point $\vec{x} = \{x_i\}_{i=1}^g$, define $B(\vec{x}) = \{\beta_j | \exists x_i \in \beta_j \text{ and } j \leq 2k\}$ to be those β-arcs occurring in $\vec{x}$.

Lemma 4.3.1 *Suppose that $(O, B(\vec{x}))$ is admissible. Let $A \in \pi_2(\vec{x}, \vec{y})$, and $u \in \mathcal{M}^{A,O}$. Then for each α_i (respectively β_j) the restriction of $\pi_\mathbb{R} \circ u$ to $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ (respectively $u^{-1}(\beta_j \times \{0\} \times \mathbb{R})$) is bijective (respectively injective).*

Proof We know that near $-\infty$, u is close to the g-tuple of strips $\vec{x} \times [0,1] \times \mathbb{R}$. It follows that $\pi_\mathbb{D} \circ u$ is a g-fold covering map. Since there are g α-circles, the statement about $u^{-1}(\alpha_i \times \{1\} \times \mathbb{R})$ follows.

Similarly, if there are m Reeb chords at east ∞ occurring at punctures q_i with $\pi_\mathbb{R} \circ u(q_i) < t$ then for $\beta_j \in B(\vec{x})_m$, $u^{-1}(\beta_j \times \{0\} \times \{t\}) \neq \emptyset$. So, since there are k

distinct β-arcs in $B(\vec{x})_m$, $u^{-1}(\beta_j \times \{0\} \times \{t\}) \neq \emptyset$ for each of the $g-k$ β-circles β_j, the result about $u^{-1}(\beta_j \times \{1\} \times \mathbb{R})$ also follows. □

4.4 Embeddedness determines χ

We recall a result of [Lip]. Let $(\Sigma, \vec{\alpha}, \vec{\beta})$ be a Heegaard diagram for a closed three-manifold, and $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ an embedded holomorphic curve. Given a point $x \in \alpha_i \cap \beta_j$ define $n_x(u)$ to be the average local multiplicity of $\pi_\Sigma \circ u$ at x. That is, identify a neighborhood of x with $\mathbb{D}$ so that α is identified with $\mathbb{R}$ and β with $i\mathbb{R}$. Then

$$n_x(u) = \frac{1}{4}\left(n_{\frac{1}{2}e^{\pi/4}}(u) + n_{\frac{1}{2}e^{3\pi/4}}(u) + n_{\frac{1}{2}e^{5\pi/4}}(u) + n_{\frac{1}{2}e^{7\pi/4}}(u)\right)$$

where $n_{\frac{1}{2}e^{k\pi/4}}(u)$ is the local multiplicity of $\pi_\Sigma \circ u$ at $\frac{1}{2}e^{k\pi/4}$. Given an intersection point $\vec{x} = \{x_i\}$ define $n_{\vec{x}}(u) = \sum_{i=1}^{g} n_{x_i}(u)$. Observe that $n_{\vec{x}}(u)$ depends only on the homology class A of u, and so we may write $n_{\vec{x}}(A)$ to mean $n_{\vec{x}}(u)$ for any representative u of A.

Given a Riemann surface D with boundary and right-angled corners, k with internal angle $\pi/2$ and ℓ with internal angle $3\pi/2$ define the Euler measure $e(D)$ of D to be $e(D) = \chi(D) - \frac{k}{4} + \frac{\ell}{4} \in \frac{1}{4}\mathbb{Z}$. The number e arises naturally in the Gauss-Bonnet theorem: for a metric on D for which the boundary is geodesic and the corners are right angles, $e(D) = \frac{1}{2\pi}\int_D \mathrm{KdA}$ where K denotes the Gauss curvature of the metric.

Extend the definition of e linearly to formal sums of Riemann surfaces. Then, since every homology class in $\pi_2(\vec{x}, \vec{y})$ corresponds to a formal sum of connected components of $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$, there is a map $e : \pi_2(\vec{x}, \vec{y}) \to \frac{1}{4}\mathbb{Z}$ defined by $e\left(\sum a_i D_i\right) = \sum a_i e(D_i)$. (The image actually lies in $\frac{1}{2}\mathbb{Z}$ since the number of corners of a domain is always even.) If u is a map in the homology class A we may write $e(u)$ to mean $e(A)$.

The following is [Lip, Proposition 4.2]:

Proposition 4.4.1 *If $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ is an embedded holomorphic curve* *connecting $\vec{x}$ to $\vec{y}$ in the homology class A then*

$$\chi(S) = g - n_{\vec{x}}(A) - n_{\vec{y}}(A) + e(A).$$

In particular, $\chi(S)$ is determined by the homology class of u.

Three remarks are in order. The first is that the result holds for u holomorphic with respect to any (cylindrical) almost complex structure on $\Sigma \times [0,1] \times \mathbb{R}$; no genericity is required. In fact, the result holds for curves satisfying certain topological restrictions. The second is that the proof does not use the homological linear independence of the α_i (respectively β_j): it holds if the α- (respectively β-) curves are any g-tuple of disjoint embedded circles. The precise formula does depend on the fact that there are g each of α- and β-circles, but a similar formula holds in the case of ℓ α- (respectively β-) circles, $\ell \neq g$. The third remark is that for curves with j double points, a similar formula holds, by the same argument: $\chi(S) = g - n_{\vec{x}}(A) - n_{\vec{y}}(A) + e(A) + 2j$. So,

Proposition 4.4.2 *Let Σ be a closed Riemann surface, $\alpha_1, \cdots, \alpha_k$ (respectively $\beta_1, \cdots, \beta_k$) pairwise disjoint embedded circles in Σ, $\vec{x} = \{x_i \in \alpha_i \cap \beta_{\sigma(i)}\}_{i=1}^k$ and $\vec{y} = \{y_i \in \alpha_i \cap \beta_{\sigma'(i)}\}_{i=1}^k$ two k-tuples of Reeb chords in $(\Sigma \times [0,1] \times \mathbb{R}, (\alpha_1 \cup \cdots \cup \alpha_k) \times \{1\} \times \mathbb{R}, (\beta_1 \cup \cdots \cup \beta_k) \times \{0\} \times \mathbb{R})$. For $i = 1, 2$ let*

$$u_i : (S_i, \partial S_i) \to (\Sigma \times [0,1] \times \mathbb{R}, [(\alpha_1 \cup \cdots \cup \alpha_k) \times \{1\} \times \mathbb{R}] \cup [(\beta_1 \cup \cdots \cup \beta_k) \times \{0\} \times \mathbb{R}])$$

be holomorphic curves with respect to some cylindrical almost complex structure on $\Sigma \times [0,1] \times \mathbb{R}$ with the same number of double points. If u_1 and u_2 are homologous then $\chi(S_1) = \chi(S_2)$.

We next deduce a similar result for curves in a Heegaard diagram with boundary from the closed case.

Proposition 4.4.3 *Let $(\Sigma, \vec{\alpha}, \vec{\beta})$ be a Heegaard diagram with boundary,*

$$u_i : (S_i, \partial S_i) \to \Big(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta} \times \{0\} \times \mathbb{R})\Big),$$

$i = 1, 2$, embedded holomorphic curves connecting $\vec{x}$ to $\vec{y}$ in $\mathcal{M}^{A,O}$. Assume u_1 and u_2 are transversely cut-out by the $\overline{\partial}$-equation. Then $\chi(S_1) = \chi(S_2)$.

Proof Roughly, we will construct a Heegaard diagram with boundary $(\Sigma', \vec{\alpha}', \vec{\beta}')$ so that Σ and Σ' can be glued together to give a closed Heegaard diagram $\Sigma \cup_\partial \Sigma'$, and a holomorphic curve $u' : S' \to \Sigma \times [0,1] \times \mathbb{R}$ so that u' can be glued to each u_i to give a holomorphic curve $u_i \cup u' : S_i \cup_\partial S' \to (\Sigma \cup_\partial \Sigma') \times [0,1] \times \mathbb{R}$. The result then follows from Proposition 4.4.1 or 4.4.2. (Actually, we will first perturb the u_i slightly, and will be forced to use Proposition 4.4.2.)

Let n_{β_i} denote the number of times β_i occurs as β_{up} or β_{down} of a Reeb chord in O. For convenience, we will assume $n_{\beta_i} > 0$ for all i; the general case is a simple adaptation of our argument. For $i = 1, \cdots, k$, let $\{\beta_{i,j}\}_{j=1}^{n_{\beta_i}}$ be n_{β_i} parallel (disjoint) copies of β_i, in a small tubular neighborhood of β_i. Let $\vec{\beta}_{\mathrm{many}} = \{\beta_{1,1}, \cdots, \beta_{1,n_{\beta_i}}, \beta_{2,1}, \cdots, \beta_{2k,n_{\beta_{2k}}}, \beta_{2k+1}, \cdots, \beta_{g+k}\}$. Deform u_i to a map

$$u_{i,\mathrm{new}} : (S_i, \partial S_i) \to \Big(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}_{\mathrm{many}} \times \{0\} \times \mathbb{R}\}\Big)$$

as follows. There are n_{β_i} different arcs $b_{j,1}, \cdots, b_{j,n_{\beta_i}}$ of ∂S mapped by u_i to $\beta_j \times \{0\} \times \mathbb{R}$. We choose the ordering of the $b_{j,k}$ so that $\pi_{\mathbb{R}} \circ u_i(b_{j,k}) < \pi_{\mathbb{R}} \circ u(b_{j,k+1})$ for all j, k. (Lemma 4.3.1 is used here.) Then $u_{i,\mathrm{new}}$ is a small deformation of u_i so that $u_{i,\mathrm{new}}(b_{j,k}) \in \beta_{j,k} \times \{0\} \times \mathbb{R}$.

If u_i is transversely cut out then one can choose $u_{i,\mathrm{many}}$ to be holomorphic.

Let $(\Sigma', \vec{\alpha}', \vec{\beta}')$ be a Heegaard diagram with boundary with the following properties.

- In a collar neighborhood of $\partial\Sigma'$, $(\Sigma', \vec{\alpha}', \vec{\beta}')$ is diffeomorphic to a collar neighborhood of $\partial\Sigma$ with the opposite orientation. (The collar neighborhood of $\partial\Sigma$ should be chosen small enough not to intersect $\vec{\alpha}$.)

- Near $\partial\Sigma'$ there is an α-circle α_1 and for each Reeb chord γ, points $x_{\gamma,\mathrm{up}} \in \alpha_1 \cap \beta_{\mathrm{up}}(\gamma)$ and $x_{\gamma,\mathrm{down}} \in \alpha_1 \cap \beta_{\mathrm{down}}(\gamma)$ so that there is a holomorphic disk D_γ in $\Sigma' \times [0,1] \times \mathbb{R}$ asymptotic to $x_{\gamma,\mathrm{down}}$ at $-\infty$, $x_{\gamma,\mathrm{up}}$ at $+\infty$ and γ at east infinity. (The various $x_{\gamma,\mathrm{down}}$ and $x_{\gamma,\mathrm{up}}$ need not all be distinct.)

Such $(\Sigma', \vec{\alpha}', \vec{\beta}')$ are easy to produce; see Figure 4.1.

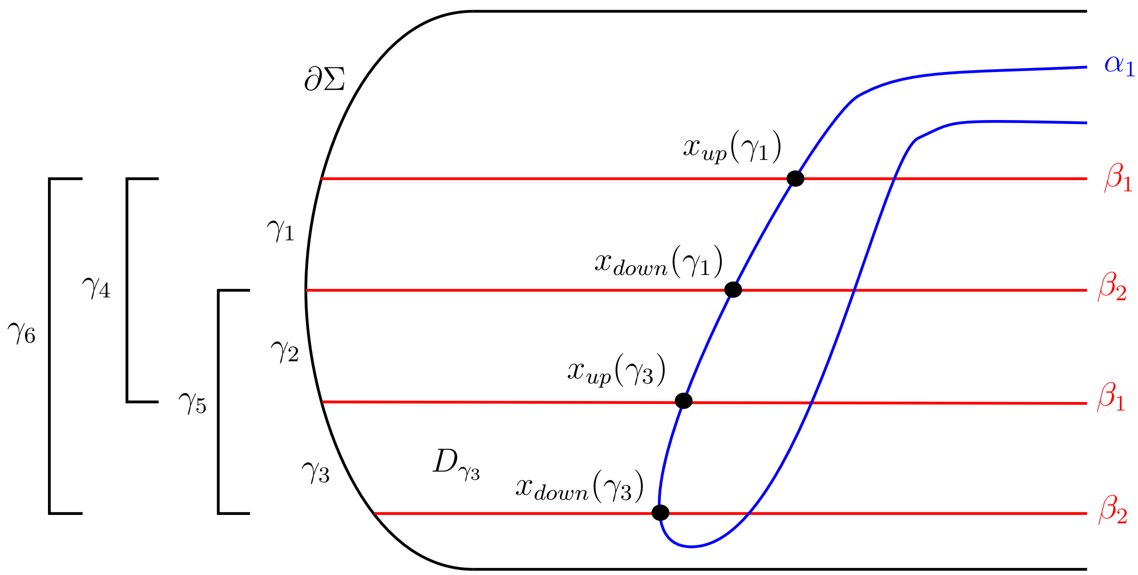


Figure 4.1: A possible $(\Sigma', \vec{\alpha}', \vec{\beta}')$. Observe $x_{\text{up}}(\gamma_1) = x_{\text{down}}(\gamma_2) = x_{\text{down}}(\gamma_5)$, and so on.

Let $\vec{\alpha}'_{\text{many}}$ be the collection of circles obtained by replacing α_1 with $|O|$ parallel copies $\alpha_{1,j}$ of α_1. Let $\vec{\beta}'_{\text{many}}$ be the collection of arcs and circles obtained by replacing β'_i with n_{β_i} parallel copies $\beta'_{i,j}$ of β'_i. Each point $x_{\gamma,\text{up}}$ corresponds to a point $x_{\gamma,\text{up},j,k} \in \alpha_{1,j} \cap \beta_{\text{up}}(\gamma)_k$ (and similarly for each $x_{\gamma,\text{down}}$), and each disk D_γ to disks $D_{\gamma,j,k,k'}$ asymptotic to $x_{\gamma,\text{down},j,k}$ at $-\infty$ and $x_{\gamma,\text{up},j,k'}$ at $+\infty$.

For a suitable union u^O of disks $D_{\gamma,j,k,k'}$, depending only on $\vec{x}$ and O, the ordered list of Reeb chords (between the $\beta_{\text{many}} \cap \partial\Sigma'$) of u^O at east ∞ is exactly the ordered list of Reeb chords of $u_{i,\text{new}}$ at east ∞. Observe also that u^O is transversely cut-out by $\overline{\partial}$.

Observe that

$$\left(\Sigma' \times [0,1] \times \mathbb{R}, (\vec{\alpha}'_{\text{many}} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}'_{\text{many}} \times \{0\} \times \mathbb{R})\right)$$

can be glued to

$$\left(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}_{\text{many}} \times \{0\} \times \mathbb{R})\right)$$

to give

$$\left((\Sigma \natural \Sigma') \times [0,1] \times \mathbb{R}, ((\vec{\alpha} \cup \vec{\alpha}'_{\text{many}}) \times \{1\} \times \mathbb{R}) \cup ((\vec{\beta}_{\text{many}} \natural \vec{\beta}'_{\text{many}}) \times \{0\} \times \mathbb{R})\right).$$

Here, $\Sigma \natural \Sigma'$ is a closed surface and $\vec{\alpha} \cup \vec{\alpha}'_{\text{many}}$ (respectively $\vec{\beta}_{\text{many}} \natural \vec{\beta}'_{\text{many}}$) a union of pairwise disjoint simple closed curves in $\Sigma \natural \Sigma'$. Further, it follows that $u_{i,\text{new}}$ can be glued to u^O ($i = 1, 2$) to give a holomorphic curve $u_{i,\text{new}} \natural u^O$ in this glued manifold. Observe that $u_{1,\text{new}} \natural u^O$ and $u_{2,\text{new}} \natural u^O$ have the same number of double points (all of which correspond to double points of u^O). It follows that the Euler characteristics of the sources of $u_{i,\text{new}} \natural u^O$ agree.

The Euler characteristic of the source of $u_{i,\text{new}} \natural u^O$ is exactly $\chi(S_i)$. It follows that $\chi(S_1) = \chi(S_2)$, as desired. □

Remark. The assumption that the u_i were transversely cut-out was convenient for the proof, but not essential. Indeed, there are much softer requirements than holomorphicity in which the result of Proposition 4.4.3 holds; compare [Lip, Lemma 4.1]. The more general result, however, requires considerably more fuss, and for us is unnecessary.

4.5 A formula for the index

In this section we derive a formula for the expected dimension of the moduli spaces $\mathcal{M}^{A,O}$ which extends the formula [Lip, Formula (6), Section 4.1] in the closed case. Before doing so we introduce another 'capping' operation.

For the moment, view Σ as a surface with boundary. Let Σ^{cap} denote the result of gluing a collar $\partial\Sigma \times [0, \varepsilon)$ to $\partial\Sigma$. Inside $(\partial\Sigma \backslash \mathfrak{z}) \times [0, \varepsilon)$, choose arcs connecting every pair of (ends of) β-arcs, as shown in Figure 4.2; call the new arcs $\vec{\beta}_0^{\text{cap}}$. Call the resulting β-arcs in Σ^{cap} $\vec{\beta}^{\text{cap}}$. (This "caps-off" the Reeb chords at east infinity.) We call the result the capped Heegaard diagram. For an appropriate choice of almost complex structure on $\Sigma^{\text{cap}} \times [0,1] \times \mathbb{R}$, stretching the neck along $\partial\Sigma \times [0,1] \times \mathbb{R}$ gives a chain of symplectic manifolds $(\Sigma \times [0,1] \times \mathbb{R}, \partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R})$.

For each Reeb chord γ_i there is a holomorphic disk D_γ in $(\partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R}, \vec{\beta}_0^{\text{cap}} \times \{0\} \times \mathbb{R})$ asymptotic to γ_i. Given a map $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ consistent with a list of Reeb chords O one can glue (really, "preglue") u to $\bigcup_{\gamma \in O} D_\gamma$ in an obvious way to obtain a map $\text{cap}(u) : \text{cap}(S) \to \Sigma^{\text{cap}} \times [0,1] \times \mathbb{R}$.

Recall from Section 4.4 the definition of the Euler measure, e. Extend e to domains going out to east infinity by viewing Σ as a surface with boundary so that the β-arcs meet $\partial\Sigma$ at right angles.

Fix a homology class $A \in \pi_2(\vec{x}, \vec{y})$, a two-level ordered list of Reeb chords O such that $(O, B(\vec{x}))$ is admissible, a decorated source $S^\diamond$ connecting $\vec{x}$ to $\vec{y}$, and a map $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ in the homology class A respecting the decorations of $S^\diamond$ and consistent with O. Let $|O|$ denote the number of Reeb chords in O and $\varepsilon(O)$ the number of $<_\varepsilon$'s occurring in O.

Proposition 4.5.1 *The expected dimension of the moduli space of holomorphic maps $S \to \Sigma \times [0,1] \times \mathbb{R}$ in the homology class A respecting the decorations of $S^\diamond$ and consistent with O is*

$$g - \chi(S) + 2e(A) + |O| - \varepsilon(O).$$

Proof First, suppose that $\varepsilon(O) = 0$. Recall ([Lip, Section 4.1, Formula (6)]) that in the closed case the index at a map $u : S \to \Sigma \times [0,1] \times \mathbb{R}$ is given by $g - \chi(S) + 2e(A)$. We deduce our formula from the closed one via the capping construction; the reader may find the details easier to produce than to read.

The closed formula applies to the linearization of the $\overline{\partial}$-operator at $\text{cap}(u)$, giving $\text{ind}(\text{cap}(u)) = g - \chi(\text{cap}(S)) + 2e(\text{cap}(A)) = g - \chi(S) + 2e(\text{cap}(A))$. Observe that the index of $D\overline{\partial}$ at D_γ is 1. It follows that the index at u is

$$\text{ind}(u) = \ g - \chi(S) + 2\left(e(A) + |O|/2\right) - |O| + |O|,$$

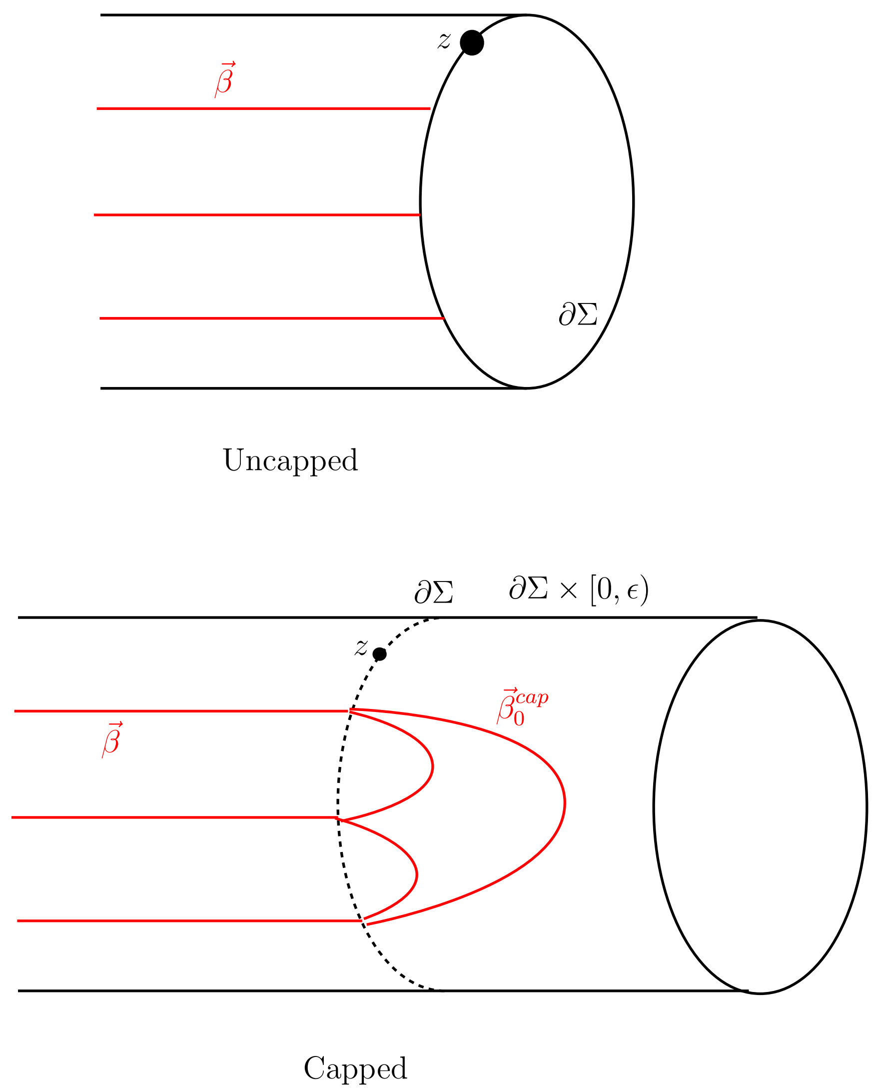


Figure 4.2: The capping operation.

where the $|O|/2$ comes from the effect of capping on the Euler measure, the $-|O|$ from the fact that $\mathrm{ind}(D_\gamma) = 1$, and the $+|O|$ comes from the matching conditions at the punctures (because of the Morse-Bott nature of the gluing). This proves the proposition in the case $\varepsilon(O) = 0$.

Now suppose $\varepsilon(O) > 0$. Write $\varepsilon(O) = \varepsilon_1 + \varepsilon_2$ where $\varepsilon_1 = \sum_{\text{punctures } q} k(q) - 1$, so ε_1 is the number of $<_\varepsilon$'s corresponding to partitions of punctures of S and ε_2 the number of constraints on different punctures of S. Then, we have

$$\mathrm{ind}(u) = g - \chi(S) + 2\left(e(A) + \frac{|O| - \varepsilon_1}{2}\right) - (|O| - \varepsilon_1) + (|O| - \varepsilon_1) - \varepsilon_2$$

where most of the terms are as before, the $-\varepsilon_1$ terms corresponds to the fact that fewer punctures are being capped, and the $-\varepsilon_2$ comes from the fact that we are imposing ε_2 point constraints in the $\mathbb{R}$-direction on the punctures. □

Remark. Proposition 4.5.1 could also have been proved by adapting the gluing argument from Section 4.4. In that argument, the computation for $\varepsilon(O) = 0$ would be $\mathrm{ind}(O) = g + |O| - \chi(S) + 2e(A)$ where the $+|O|$ comes from the fact that the glued curves have $g + |O|$ ends at $\pm\infty$.

The following is immediate from Propositions 4.4.3 and 4.5.1.

Corollary 4.5.2 *The expected dimension of the space of embedded holomorphic curves in the homology class A consistent with O near a curve u depends only on A and O. (We will denote this expected dimension* $\mathrm{ind}(A, O)$*.)*

4.6 Restrictions on codimension 1 degenerations

This section brings together most of the technical results proved so far to analyze completely the codimension 1 degenerations of our moduli spaces. Proposition 4.6.1 is the culmination of the analytic work in this paper; almost all of the rest of the paper consists of interpreting this result algebraically.

By a *join curve* we mean a map $S \to (\partial\Sigma) \times \mathbb{R} \times [0, 1] \times \mathbb{R}$ with one component a disk with three boundary punctures, one mapped to east ∞ and two mapped to

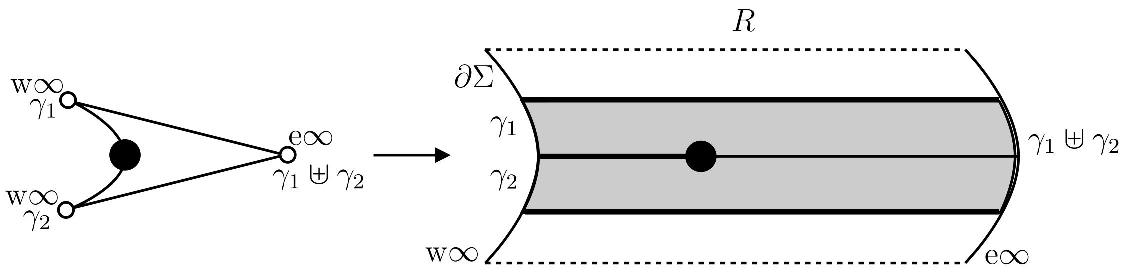


Figure 4.3: A *join component* and its image in $\partial\Sigma \times \mathbb{R}$.

west ∞, and all other components disks with two boundary punctures (one mapped to east ∞ and one mapped to west ∞). See Figure 4.3. Such curves will correspond to the operation join defined in Section 3.1. We may also speak of *backwards join curves*, which are the reflections in the first $\mathbb{R}$-factor of join curves.

We call a holomorphic curve $S \to (\partial\Sigma) \times \mathbb{R} \times [0,1] \times \mathbb{R}$ *trivial* if it is a disjoint union of disks with two boundary punctures each. Such curves are not stable, and hence do not appear in limits of sequences of holomorphic curves.

Proposition 4.6.1 *Fix intersection points $\vec{x}$, $\vec{y}$, a homology class $A \in \pi_2(\vec{x}, \vec{y})$, and an ordered set of Reeb chords Γ. Assume* $\mathrm{ind}(A, \Gamma) = 2$. *Then, for a generic choice of J satisfying **(J1)**–**(J4)**, the boundary of $\mathcal{M}^{A,\Gamma}$ consists of the following pieces:*

1. *Holomorphic buildings, split at $\pm\infty$. That is,*

$$\bigcup_{\substack{A_1 + A_2 = A \\ \Gamma_1 < \Gamma_2 = \Gamma}} \mathcal{M}^{A_1,\Gamma_1} \times \mathcal{M}^{A_2,\Gamma_2}.$$

2. *Holomorphic buildings split at east infinity, where the level at east infinity is a join curve. That is, $\bigcup \mathcal{M}^{A,\Gamma'}$ where Γ' is obtained from Γ by replacing some $\gamma = \gamma_a \uplus \gamma_b$ by $\gamma_b <_\varepsilon \gamma_a$, and Γ' is admissible.*

3. *A single collapse in the ordering. That is, $\bigcup \mathcal{M}^{A,\Gamma'}$ where the union is over Γ' obtained from Γ by replacing one $<$ by a $<_\varepsilon$.*

Proof The proof is in several steps. By Proposition 4.2.1, any sequence $\{u^i\}_{i=1}^\infty$ of curves in $\mathcal{M}^{A,\Gamma}$ has a subsequence converging to some holomorphic comb $\{u_i^\infty, v_{i,j}^\infty\}$.

Using Proposition 4.4.3, we will show first that all of the $v_{i,j}^\infty$ must be topological disks, and $\mathrm{ev}_e : \mathcal{M}(v_{i,j}^\infty) \to \mathbb{R}^{N_{i,j}}$ and $\mathrm{ev}_w : \mathcal{M}(v_{i,j+1}^\infty) \to \mathbb{R}^{N_{i,j}}$ are transverse. It then follows from Lemma 4.1.2 that the linearized $\overline{\partial}$-map is surjective at the $v_{i,j}^\infty$. We know by Proposition 4.1.1 that the linearized $\overline{\partial}$-map is surjective at the u_i^∞. For a generic choice of J, $\mathrm{ev} : \mathcal{M}(u_i^\infty) \to \mathbb{R}^{N_i}$ and $\mathrm{ev}_w : \mathcal{M}(v_{i,0}) \to \mathbb{R}^{N_i}$ are transverse. The result will then follows from Proposition 4.2.2, an explicit computation of the moduli spaces of disks in $(\partial\Sigma) \times \mathbb{R} \times [0,1] \times \mathbb{R}$, and a few words about why cusp degenerations are impossible.

We begin by deducing from the fact that A, Γ determine the Euler characteristic of the source S that all of the $v_{i,j}^\infty$ are unions of disks and, further, that if for each i one glues the $v_{i,j}^\infty$ (in the obvious way) the components are still unions of disks. Let v_i denote a curve obtained by gluing the $v_{i,j}^\infty$ at east/west ∞. There is a curve v_i' in $\partial\Sigma \times [0,1] \times \mathbb{R}$ all of whose components are disks and with the same asymptotics as v_i at east and west ∞. By Lemma 4.1.2 and Proposition 4.2.2 can glue the v_i to the u_i^∞ at east/west ∞, and glue the resulting curves at $\pm\infty$, to give a new curve in $\mathcal{M}^{A,\Gamma}$. If the components of the v_i were not all disks then the Euler characteristic of the new curve will not be the same as that of the old ones, contradicting Proposition 4.4.3.

Note in particular that this means that for each component C of $v_{i,j}$ and C' of $v_{i,j+1}$, the west ∞ of C' meets the east ∞ of C in at most one Reeb chord. It follows that the evaluation maps ev_w from $\mathcal{M}(v_{i,j+1})$ and ev_e from $\mathcal{M}(v_{i,j})$ are transverse.

Now, for a map $v : S \to \partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R}$, the dimension of $\mathcal{M}(v)$ is

$$-2 + (\#\text{components of v}) + 2(\#\text{of branch points of} \pi_\Sigma \circ v)$$

(The -2 comes from the fact that curves in $\partial\Sigma \times \mathbb{R} \times [0,1] \times \mathbb{R}$ are only defined up to translation in the two $\mathbb{R}$-factors. Boundary branch points count for 1/2.) Observe also that $\mathrm{ev}_w(v)$ is contained in a subset of codimension

$$(\#\text{of Reeb chords of v at east} \infty) - (\#\text{of components of} v).$$

Using Lemma 4.1.2 and Proposition 4.2.2 and the fact that the $v_{i,j}^\infty$ are unions of disk components, it follows that at most one of the $v_{i,j}^{\infty}$ is nontrivial, and that one is either a join curve or a backwards join curve.

It also follows that if any $v_{i,j}^{\infty}$ is nontrivial then the comb has only one story, and that the comb never has more than two stories.

Observe that the appearance of a backwards join curve corresponds to a collapse of Γ (item 3 above). If $\gamma_i < \gamma_j$ appears in Γ with $\beta_{\text{up}}(\gamma_i) = \beta_{\text{down}}(\gamma_j)$ this is the only way a collapse can occur. If $\beta_{\text{up}}(\gamma_i) \neq \beta_{\text{down}}(\gamma_j)$ then collapses can still occur but do not split off backwards join curves.

A sequence splitting into two stories corresponds to item 1.

A sequence splitting off a join curve corresponds to item 2. Forgetting the new component at east ∞ gives an element of $\mathcal{M}^{A,\Gamma'}$ where Γ' is obtained from Γ by replacing the east infinity of the join component $\gamma = \gamma_a \uplus \gamma_b$ by its west infinity $\gamma_b <_\varepsilon \gamma_a$. For this splitting to occur, by Lemma 4.3.1, the new ordered list of Reeb chords Γ' must be admissible.

Bubbling of disks or spheres is ruled out since $\pi_2(\Sigma) = \pi_2(\Sigma, \vec{\alpha}) = \pi_2(\Sigma, \vec{\beta}) = 0$. Deligne-Mumford degenerations in the interior of S are codimension 2.Cusp degenerations (not corresponding to splitting curves at $\pm\infty$ of east ∞) could occur. This is, however, ruled-out by Lemma 4.3.1. (Compare [Lip, Proposition 7.1].) □

Chapter 5

Heegaard-Floer invariants of bordered 3-manifolds

5.1 Definition of the Heegaard-Floer differential module

Given a pointed Heegaard diagram with boundary $(\Sigma, \alpha, \beta, \mathfrak{z})$, let $\mathcal{A}$ denote the differential algebra associated to the boundary of Σ. To the Heegaard diagram with boundary we will associate a left differential $\mathcal{A}$-module CF. In later sections we will show that CF depends only on the bordered three-manifold specified by $(\Sigma, \alpha, \beta, \mathfrak{z})$.

Let CF_0 be the free $\mathbb{F}_2$-module on the intersection points in Σ. Then CF_0 is naturally a left $\mathcal{A}_0$-module: define

$$(\emptyset, B) \cdot \vec{x} = \begin{cases} 0 & \text{if} B \neq B(\vec{x}) \\ \vec{x} & \text{if} B = B(\vec{x}) \end{cases} .$$

Let $\mathrm{CF} = \mathcal{A} \otimes_{\mathcal{A}_0} \mathrm{CF}_0$, a left $\mathcal{A}$-module. Observe that CF is spanned by elements of the form $(O, B) \otimes \vec{x}$ where $B(\vec{x}) = B_{|O|}$. We define a map $d : \mathrm{CF}_0 \to \mathrm{CF}$ by

$$d\vec{x} = \sum_{\vec{y}} \sum_{\substack{A \in \pi_2(\vec{x}, \vec{y}) \\ \mathrm{ind}(A, O) = 1}} \left(\#\mathcal{M}^{A,O}\right) (O, B(x)) \otimes \vec{y}.$$

Extend d to a map $d : \mathrm{CF} \to \mathrm{CF}$ by requiring that d satisfy the Leibniz rule, i.e., $d\left((O,B)\otimes\vec{x}\right) = (d(O,B))\otimes\vec{x} + (O,B)\otimes d\vec{x}$.

Lemma 5.1.1 *If the pointed Heegaard diagram with boundary $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ is weakly admissible then the sum defining $d\vec{x}$ is finite.*

Proof This follows from Corollary 2.7.3 and the fact that for $q \in \Sigma$, $\{q\}\times[0,1]\times\mathbb{R}$ is J-holomorphic. □

Proposition 5.1.2 $d^2 = 0$.

Proof Since

$$d^2\left[(O,B)\vec{x}\right] = \left[d^2(O,B)\right]\vec{x} + 2\left[d(O,B)\right]\left[d\vec{x}\right] + (O,B)[d^2\vec{x}],$$

it suffices to show that $d^2\vec{x} = 0$ for $\vec{x}$ an intersection point. This will follow from Proposition 4.6.1.

$$\begin{aligned} d^2\vec{x} \;=\; & \sum_{\vec{z}}\sum_{\vec{y}} \sum_{\substack{A_1\in\pi_2(\vec{x},\vec{y})\\ A_2\in\pi_2(\vec{y},\vec{z})\\ \mathrm{ind}(A_i,O_i)=1}} \#\left(\mathcal{M}^{A_1,O_1}\times\mathcal{M}^{A_2,O_2}\right)\left[(O_1O_2, B(x))\vec{z}\right] \\ & + \sum_{\vec{z}} \sum_{\substack{A\in\pi_2(\vec{x},\vec{z})\\ \mathrm{ind}(A,O)=1}} \left(\#\mathcal{M}^{A,O}(\vec{x},\vec{z})\right)\left[\left(d(O,B(x))\right)\vec{z}\right]. \end{aligned}$$

So, we need to check that for given $\vec{z}$ and O with $(O, B(\vec{x}))$ admissible,

$$\sum_{\substack{A\in\pi_2(\vec{x},\vec{z})\\ O\in\mathrm{decol}(O')\cup\mathrm{join}(O')\\ \mathrm{ind}(A,O')=1}} \left(\#\mathcal{M}^{A,O'}(\vec{x},\vec{z})\right) + \sum_{\vec{y}} \sum_{\substack{A_1\in\pi_2(\vec{x},\vec{y})\\ A_2\in\pi_2(\vec{y},\vec{z})\\ O_1O_2=O\\ \mathrm{ind}(A_i,O_i)=1}} \#\left(\mathcal{M}^{A_1,O_1}\times\mathcal{M}^{A_2,O_2}\right) = 0.$$

To see this, observe

$$\sum_{\substack{A\in\pi_2(\vec{x},\vec{z})\\ O\in\mathrm{decol}(O')\cup\mathrm{join}(O')\\ \mathrm{ind}(A,O')=1}} \Big(\#\mathcal{M}^{A,O'}(\vec{x},\vec{z})\Big) + \sum_{\vec{y}} \sum_{\substack{A_1\in\pi_2(\vec{x},\vec{y})\\ A_2\in\pi_2(\vec{y},\vec{z})\\ O_1O_2=O\\ \mathrm{ind}(A_i,O_i)=1}} \#\left(\mathcal{M}^{A_1,O_1}\times\mathcal{M}^{A_2,O_2}\right)$$
$$= \sum_{\substack{A\in\pi_2(\vec{x},\vec{y})\\ \mathrm{ind}(A,O)=2}} \#\left(\partial\overline{\mathcal{M}}^{A,O}(\vec{x},\vec{z})\right)$$
$$= 0.$$

Here, the first equality follows from Proposition 4.6.1. □

Lemma 5.1.3 *The differential module* CF *decomposes, as a differential module, as a direct sum over* $\mathrm{Spin}^{\mathbb{C}}$*-structures on* Y*,*

$$\mathrm{CF} = \oplus_{\mathfrak{s}\in\mathrm{Spin}^{\mathbb{C}}(Y)} \mathrm{CF}^{\mathfrak{s}}.$$

Proof Clear from Lemmas 2.6.2 and 2.6.3. □

Observe that the filtration H on $\mathcal{A}$ by total homology class induces a filtration on CF. Let $\mathrm{CF}^{\wedge} = \widehat{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{CF}$ denote the completion of CF with respect to the filtration H by the total homology class of the Reeb chords at east infinity. For some computations it is easier to work with $\mathrm{CF}^{\wedge}$ than with CF.

Remark. As in traditional Heegaard-Floer homology, there are twisted versions of CF, keeping track of the homology classes A. The totally twisted version, from which all other versions can be reconstructed, is defined as follows. Fix $s \in \mathrm{Spin}^{\mathbb{C}}(Y)$. Let $\mathcal{A}^{\mathrm{tw}} = \mathbb{F}_2[H_2(Y,\partial Y)] \otimes_{\mathbb{F}_2} \mathcal{A}$, with differential induced by the differential on $\mathcal{A}$, and as a module let $\mathrm{CF}^s_{\mathrm{tw}} = \mathcal{A}^{\mathrm{tw}} \otimes_{\mathcal{A}} \mathrm{CF}$. To define a differential on $\mathrm{CF}^s_{\mathrm{tw}}$, fix an intersection point $\vec{x}_0 \in \mathrm{CF}^s_0$. For each intersection point $\vec{x} \in \mathrm{CF}^s_0$, choose an element $A_{\vec{x}} \in \pi_2(\vec{x}_0, \vec{x})$. These choices identify $\pi_2(\vec{x},\vec{y})$ with $\pi_2(\vec{x}_0,\vec{x}_0)$, which in turn is identified with $H_2(Y,\partial Y)$ (Lemma 2.6.1). Now define for $\vec{x} \in \mathrm{CF}^s_0$, define

$d_{\mathrm{tw}}\vec{x} \in \mathrm{CF}^s_{\mathrm{tw}}$ by

$$d_{\mathrm{tw}}\vec{x} = \sum_{\vec{y}} \sum_{\substack{A \in \pi_2(\vec{x},\vec{y}) \\ \mathrm{ind}(A,O)=1}} \left(\#\mathcal{M}^{A,O}\right) [(A) \otimes (O, B(x))] \otimes \vec{y}$$

where (A) denotes the image of A in $H_2(Y, \partial Y)$ under the identification induced by the $A_{\vec{x}}$. Extend this definition to a map $\mathrm{CF}^s_{\mathrm{tw}} \to \mathrm{CF}^s_{\mathrm{tw}}$.

Note that since both $\mathcal{A}$ and $H_2(Y, \partial Y)$ keep track of the total homology class of the Reeb chords of a domain at east ∞, there is some redundancy in $\mathrm{CF}^s_{\mathrm{tw}}$.

One can check that this chain complex is independent of the choices of the $A_{\vec{x}}$. Further, the proof of invariance of CF given below extends to prove that $\mathrm{CF}^s_{\mathrm{tw}}$ depends only on (Y, s). That said, we will not discuss $\mathrm{CF}^s_{\mathrm{tw}}$ further in this thesis.

5.2 Remarks on the convergence of spectral sequences

To compute the homology of CF we would like to exploit the filtrations by total homology class and number of Reeb chords. Unfortunately, convergence of these spectral sequences is somewhat subtle. To illustrate, consider the following toy example, which captures the spirit of the issues involved.

Consider the differential algebra $\mathbb{F}_2[x,y]/(x^2)$ with differential specified by $d(xy^k) = y^{k+1} + y^{k+2}$ and $d(y^k) = 0$. On the one hand, the homology of $(\mathbb{F}_2[x,y]/(x^2), d)$ is $\mathbb{F}_2$, generated by y. On the other hand, there is a filtration $F^0 \supset F^1 \supset \cdots$ of $\mathbb{F}_2[x,y]/(x^2)$ where F^s is generated by all monomials of total degree at least s. The homology of the associated graded module is zero, so the spectral sequence associated to the filtration F^s converges to zero, not $\mathbb{F}_2$. Observe, however, that if one completes $\mathbb{F}_2[x,y]/(x^2)$ with respect to F^s then the homology is also zero: the completion is just $(\mathbb{F}_2[x]/x^2)[\![y]\!]$, and in $(\mathbb{F}_2[x]/x^2)[\![y]\!]$, $y = d[(1 + y + y^2 + \cdots)x]$.

The convergence results we need are contained in [Boa99].

Definition 5.2.1 *Let (C, d) denote a differential module and $\cdots \supset F^{s-1} \supset F^s \supset F^{s+1} \supset \cdots$ a decreasing filtration of C.*

- *The filtration F is exhaustive if $\bigcup_{s=-\infty}^{\infty} F^s = C$.*
- *The filtration F is left-boring if for some s_0, $F^s = F^{s_0}$ for $s < s_0$.*

(The terminology *left-boring* is not standard.)

Proposition 5.2.2 *Let (C, d) be a differential module and F an exhaustive, left-boring filtration of (C, d). Let $(\hat{C}, d)$ denote the completion of (C, d) with respect to F. Let E_r^s denote the s^{th} graded part in the r^{th} page of the spectral sequence associated to (C, d, F), and $d_r^s : E_r^s \to E_r^{s+r}$ the differential on the r^{th} page. Assume that for each s there are finitely many r for which d_r^s is nonzero. Then the spectral sequence E_r^s converges strongly to the homology of $(\hat{C}, d)$.*

Proof Unwinding the definitions, this is immediate from [Boa99, Theorem 9.3] and [Boa99, Theorem 7.1]. (See also the remark following [Boa99, Theorem 7.1].) □

Corollary 5.2.3 *Let (C, d) be a differential module and F an exhaustive, left-boring filtration of (C, d). Let $(\hat{C}, d)$ denote the completion of (C, d) with respect to F. Suppose that the spectral sequence associated to (C, d, F) collapses at some stage. Then the spectral sequence converges strongly to the homology of $(\hat{C}, d)$.*

5.3 Examples

Consider first the Heegaard diagram with boundary $\mathcal{H}_0$ shown in Figure 5.1. (Here, opposite sides of the square are identified and the corners of the square deleted.) Observe that this diagram is weakly admissible as in Definition 2.7.1. Let γ_1, γ_2 and γ_3 denote the irreducible Reeb chords indicated in the figure, and let $\gamma_4 = \gamma_1 \uplus \gamma_2$, $\gamma_5 = \gamma_2 \uplus \gamma_3$ and $\gamma_6 = \gamma_1 \uplus \gamma_2 \uplus \gamma_3$. There are three intersection points, r, s and t. Let D_1, D_2 and D_3 denote the three regions indicated in the figure (so, for instance, $D_2 \in \pi_2(r, s)$).

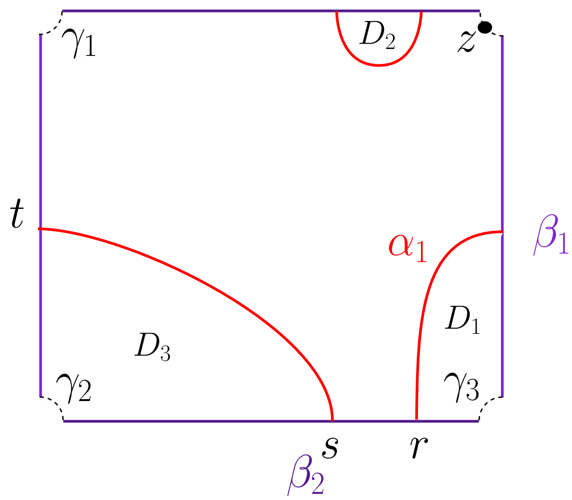


Figure 5.1: The Heegaard diagram $\mathcal{H}_0$ for $\mathbb{D}^2 \times S^1$.

The admissible pairs consisting of a single Reeb chord are (γ_1, β_1), (γ_2, β_2), (γ_3, β_1), (γ_4, β_1), (γ_5, β_2) and (γ_6, β_1). This will be important for our calculation of the homology.

We have differential

$$\begin{aligned} d(r) &= 0 \\ d(s) &= r + (\gamma_2, \beta_2)t + (\gamma_2 <_\varepsilon \gamma_3, \beta_2)r + (\gamma_5, \beta_2)r \\ d(t) &= (\gamma_3, \beta_1)r. \end{aligned}$$

We can check directly that $d^2 = 0$: certainly $d^2 r = d^2 t = 0$, while

$$d^2 s = d[r + (\gamma_2, \beta_2)t + (\gamma_2 <_\varepsilon \gamma_3, \beta_2)r] = 0 + (\gamma_2, \beta_2) \cdot (\gamma_3, \beta_1)r + (\gamma_2 < \gamma_3, \beta_2)r = 0.$$

It is easier to compute the homology of $\mathrm{CF}^\wedge(\mathcal{H}_0)$ than $\mathrm{CF}(\mathcal{H}_0)$ because for $\mathrm{CF}^\wedge(\mathcal{H}_0)$ we can use the spectral sequences associated to the filtrations by homology class and number of Reeb chords. Let $\mathrm{gr}_H(\mathrm{CF})$ denote the graded complex associated to the filtration of $\mathrm{CF}(\mathcal{H}_0)$ by the homology class of the Reeb chords at east infinity. There is a spectral sequence with E_1-term $H_*(\mathrm{gr}_H(\mathrm{CF}))$, which converges, by Proposition 5.2.2, to the homology of $\mathrm{CF}^\wedge(\mathcal{H}_0)$.

To compute $H_*\left(\mathrm{gr}_H(\mathrm{CF})\right)$, notice that, as a differential module, $H_*\left(\mathrm{gr}_H(\mathrm{CF})\right) = (\mathcal{A} \otimes_{\mathcal{A}_0} \mathbb{F}_2\langle r, s\rangle) \oplus (\mathcal{A} \otimes_{\mathcal{A}_0} \mathbb{F}_2\langle t\rangle)$. The homology of $\mathcal{A} \otimes_{\mathcal{A}_0} \mathbb{F}_2\langle t\rangle$ is just

$$\mathbb{F}_2\langle(\beta_1, \emptyset)t, (\beta_2, \gamma_2)t, (\beta_1, \gamma_5)t\rangle.$$

On each graded piece of $\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle$ there is a finite filtration by the number of Reeb chords appearing. Let $\mathrm{gr}_{\#}(\mathcal{A}\otimes\mathbb{F}_2\langle r, s\rangle)$ denote the associated graded complex; there is a spectral sequence with E_1-term $\mathrm{gr}_{\#}(\mathcal{A}\otimes\mathbb{F}_2\langle r, s\rangle)$ converging to the homology of $\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle$. On $\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)$ there is a further filtration by minus the number of $<_\varepsilon$ appearing. Let $\mathrm{gr}_\varepsilon\left(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)\right)$ denote the associated graded complex. There is a spectral sequence with E_1-term $H_*\left(\mathrm{gr}_\varepsilon\left(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)\right)\right)$ converging to $H_*\left(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)\right)$.

In $\mathrm{gr}_\varepsilon\left(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)\right)$ the internal differential of $\mathcal{A}$ has been completely eliminated, so $d\,(\Gamma r) = 0$ and $d(\Gamma s) = \Gamma r$. It follows that $H_*\left(\mathrm{gr}_\varepsilon\left(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)\right)\right) = 0$, so $H_*(\mathrm{gr}_{\#}(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle)) = 0$ so $H_*(\mathcal{A} \otimes \mathbb{F}_2\langle r, s\rangle) = 0$, so $H_*(\mathrm{gr}_H(\mathrm{CF})) = \mathbb{F}_2\langle(\beta_1, \emptyset)t, (\beta_2, \gamma_2)t, (\beta_1, \gamma_5)t\rangle$. It is now clear that there are no higher differentials in the spectral sequence converging to $H_*(\mathrm{CF}^\wedge(\mathcal{H}_0))$, so

$$H_*(\mathrm{CF}^\wedge(\mathcal{H}_0)) = \mathbb{F}_2\langle(\beta_1, \emptyset)t, (\beta_2, \gamma_2)t, (\beta_1, \gamma_5)t.\rangle$$

Next, consider the Heegaard diagram with boundary $\mathcal{H}_1$ shown in Figure 5.2. This is another Heegaard diagram for a solid torus, but with a different framing (parameterization) of the boundary. As before, let let $\gamma_4 = \gamma_1 \uplus \gamma_2$, $\gamma_5 = \gamma_2 \uplus \gamma_3$ and $\gamma_6 = \gamma_1 \uplus \gamma_2 \uplus \gamma_3$. The module $\mathrm{CF}(\mathcal{H}_1)$ is generated by $\{a, b, c, d, e, f\}$; it is easy

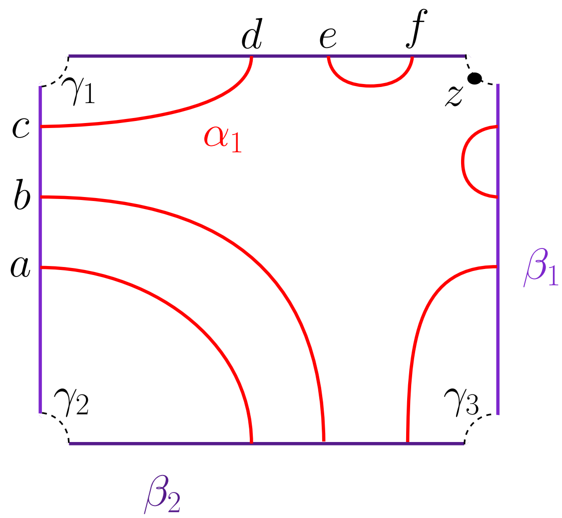


Figure 5.2: The Heegaard diagram $\mathcal{H}_1$ for $\mathbb{D}^2 \times S^1$.

to check that the differential is given by

$$\begin{aligned}
\mathrm{da} &= (\gamma_3, \beta_1)f \\
\mathrm{db} &= 0 \\
\mathrm{dc} &= b + (\gamma_1, \beta_1)d + (\gamma_4, \beta_1)a + (\gamma_1 <_\varepsilon \gamma_2, \beta_1)a + (\gamma_6, \beta_1)f \\
&\quad +(\gamma_1 <_\varepsilon \gamma_5, \beta_1)f + (\gamma_4 <_\varepsilon \gamma_3, \beta_1)f + (\gamma_1 <_\varepsilon \gamma_2 <_\varepsilon \gamma_3, \beta_1)f \\
\mathrm{dd} &= (\gamma_2, \beta_2)a + (\gamma_5, \beta_2)f + (\gamma_2 <_\varepsilon \gamma_3, \beta_2)f \\
\mathrm{de} &= f + (\gamma_2, \beta_2)b \\
\mathrm{df} &= 0.
\end{aligned}$$

Let us compute the homology of $\mathrm{CF}^\wedge(\mathcal{H}_1)$. In $\mathrm{gr}_\varepsilon(\mathrm{gr}_\#(\mathrm{gr}_H(\mathrm{CF}(\mathcal{H}_1))))$, $d(\Gamma c) = \Gamma b$ and $d(\Gamma e) = \Gamma f$; all other differentials are zero. So, the homology of $\mathrm{gr}_\varepsilon(\mathrm{gr}_\#(\mathrm{gr}_H(\mathrm{CF}(\mathcal{H}_1))))$ is generated by a and d. Next, the homology of $\mathrm{gr}_\#(\mathrm{gr}_H(\mathrm{CF}(\mathcal{H}_1)))$ is

$$\mathbb{F}_2\langle(\emptyset, \beta_1)a, (\gamma_2, \beta_2)a, (\gamma_4, \beta_1)a, (\emptyset, \beta_2)d, (\gamma_1, \beta_1)d, (\gamma_3, \beta_1)d, (\gamma_5, \beta_1)d, (\gamma_6, \beta_1)d\rangle.$$

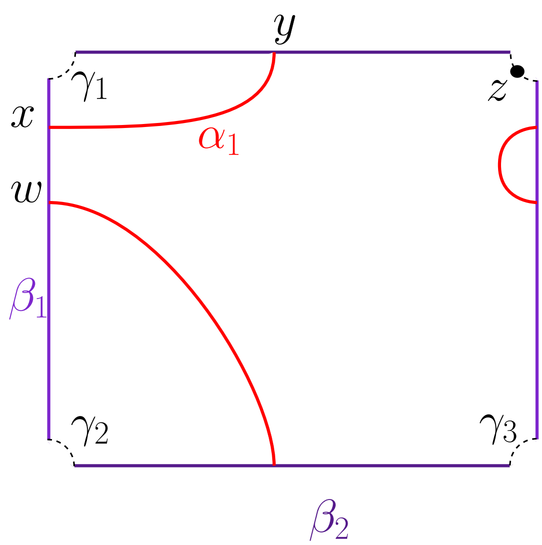


Figure 5.3: The Heegaard diagram $\mathcal{H}_\infty$ for $\mathbb{D}^2 \times S^1$.

The homology of $\mathrm{gr}_H(\mathrm{CF}(\mathcal{H}_1))$ is the same. Finally, the homology of $\mathrm{CF}(\mathcal{H}_1)$ is

$$\mathbb{F}_2\langle(\emptyset,\beta_1)a,(\gamma_{12},\beta_1)a,(\gamma_1,\beta_1)d,(\gamma_3,\beta_1)d,(\gamma_{23},\beta_1)d,(\gamma_{123},\beta_1)d\rangle.$$

since now $d((\emptyset,\beta_2)d) = (\gamma_2,\beta_2)a$. This verifies the (unremarkable) fact that $\mathrm{CF}(Y,\partial Y)$ is indeed sensitive to the parameterization of ∂Y.

(Another way to prove the same would have been to consider the change of coefficient homomorphism $\mathcal{A} \to \mathbb{F}_2$ which takes $(O,B) \to 0$ unless $O = \emptyset$, and $(\emptyset, B) \to 1$. It is obvious that $H_*(\mathbb{F}_2 \otimes_{\mathcal{A}} \mathrm{CF}(\mathcal{H}_\prime)) = \mathbb{F}_\in\langle\sqcup\rangle$, which is one-dimensional, while $H_*(\mathbb{F}_2 \otimes_{\mathcal{A}} \mathrm{CF}(\mathcal{H}_1)) = \mathbb{F}_2\langle a, d\rangle$, which is two-dimensional.)

Finally, consider the Heegaard diagram with boundary $\mathcal{H}_\infty$ shown in Figure 5.3. This is yet another Heegaard diagram for the solid torus. Here, the module $\mathrm{CF}(\mathcal{H}_\infty)$ is generated by $\{w, x, y\}$, and the differential is given by

$$\begin{aligned}
\mathrm{dw} &= 0\\
\mathrm{dx} &= w + (\gamma_1,\beta_1)y + (\gamma_4,\beta_1)w + (\gamma_1 <_\varepsilon \gamma_2, \beta_1)w\\
\mathrm{dy} &= (\gamma_2,\beta_2)w.
\end{aligned}$$

Observe that there is a short exact sequence of $\mathcal{A}$-modules

$$0 \longrightarrow \mathrm{CF}(\mathcal{H}_0) \xrightarrow{\Phi} \mathrm{CF}(\mathcal{H}_1) \xrightarrow{\Psi} \mathrm{CF}(\mathcal{H}_\infty) \longrightarrow 0 \tag{5.1}$$

where Φ and Ψ are given by

$$\begin{array}{rclrcl}
\Phi(r) & = & f & \Psi(a) & = & w \\
\Phi(s) & = & d+e & \Psi(b) & = & w \\
\Phi(t) & = & a+b & \Psi(c) & = & x \\
 & & & \Psi(d) & = & y \\
 & & & \Psi(e) & = & y \\
 & & & \Psi(f) & = & 0
\end{array}$$

It is easy to check that Φ and Ψ give chain maps.

Recall ([OS04c, Theorem 9.12]) that if Y is a 3-manifold, $K \hookrightarrow Y$ a knot, μ the meridian of K, and $\lambda \in H_1(\partial\,\mathrm{nbd}(K))$ any longitude of K (i.e., $\mu \cdot \lambda = 1$ in $H_*(\partial\,\mathrm{nbd}(K))$), there is an exact triangle relating $\widehat{\mathrm{HF}}(Y_\lambda(K))$, $\widehat{\mathrm{HF}}(Y_{\lambda+\mu}(K))$ and $\widehat{\mathrm{HF}}(Y_\mu(K)) = \widehat{\mathrm{HF}}(Y)$. (Here, $Y_\lambda(K)$ denotes surgery along K with framing λ, and so on.) This surgery exact triangle is obviously closely related to the short exact sequence (5.1) above. Indeed, if Conjecture 8.1.1 holds, then the surgery sequence for $\widehat{\mathrm{HF}}$ follows from the short exact sequence (5.1).

Chapter 6

Isotopy and stabilization invariance

We have associated to the data $\left(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z}\right)$, together with an almost complex structure J on $\Sigma \times [0,1] \times \mathbb{R}$, a filtered differential module CF over the algebra $\mathcal{A}$. In this section and the next we show that CF, up to chain homotopy equivalence, depends only on the bordered manifold $(Y, \partial Y)$ specified by $\left(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z}\right)$. The proof is a modification of the one given in [Lip, Section 9] to the relative case; the reader may find it helpful to read the proof in the closed case first.

Definition 6.0.1 *Two differential $\mathcal{A}$-modules M and N are homotopy equivalent if there are maps $\Phi : M \to N$ and $\Phi' : N \to M$ of $\mathcal{A}$-modules such that*

- *The maps Φ and Φ' are chain maps, i.e., $d \circ \Phi + \Phi \circ d = 0$ and $d \circ \Phi' + \Phi' \circ d = 0$ and*
- *There are maps $H_M : M \to M$ and $H_N : N \to N$ of $\mathcal{A}$-modules such that $\Phi' \circ \Phi = d \circ H_M + H_M \circ d$ and $\Phi \circ \Phi' = d \circ H_N + H_N \circ d$.*

Proposition 6.0.2 *Suppose that $\left(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z}, J\right)$ and $\left(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z}', J'\right)$ are related by*

- *isotopies $\vec{\alpha}_t$ and $\vec{\beta}_t$, $t \in [0,1]$, of the α- and β-arcs, constant near $\partial\Sigma$, not crossing $\mathfrak{z}$, with every intermediate Heegaard diagram weakly admissible, and*

- *a homotopy J_t of complex structures, constant near $\partial\Sigma$.*

Then $\mathrm{CF}\left(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z}, J\right)$ *and* $\mathrm{CF}\left(\Sigma, \vec{\alpha}', \vec{\beta}', \mathfrak{z}', J'\right)$ *are homotopy equivalent* $\mathcal{A}$-*modules. The homotopy equivalence respects the decomposition of* CF *into* $\mathrm{Spin}^{\mathbb{C}}$-*structures.*

As a first step in the proof, observe that we can assume the isotopies of the α- and β-arcs are Hamiltonian: any isotopy can be broken up as a sequence of Hamiltonian isotopies introducing or eliminating pairs of intersection points between α- and β-circles (i.e., *finger moves*) and a deformation of the almost complex structure; see [OS04d, Theorem 7.3] or [Lip, Section 9]. The proofs of both parts of Proposition 6.0.2 are essentially the same, so from now on, we will assume that $\vec{\alpha}_t$ and $\vec{\beta}_t$ are Hamiltonian deformations.

Fix almost complex structures J_i, $i = 1, 2$, on $\Sigma \times [0, 1] \times \mathbb{R}$ satisfying (**J1**)–(**J4**) and achieving transversality for index ≤ 1 holomorphic curves in

$$\left((\Sigma\backslash\mathfrak{z}) \times [0, 1] \times \mathbb{R}, (\vec{\alpha}_i \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}_i \times \{0\} \times \mathbb{R})\right)$$

respectively. Choose a g-tuples C_α (respectively C_β) of Lagrangian cylinders in $\Sigma \times \{1\} \times \mathbb{R}$ (respectively $\Sigma \times \{0\} \times \mathbb{R}$) agreeing with $\vec{\alpha}_0 \times \{1\} \times \mathbb{R}$ (respectively $\vec{\beta}_0 \times \{0\} \times \mathbb{R}$) near $-\infty$ and $\vec{\alpha}_1 \times \{1\} \times \mathbb{R}$ (respectively $\vec{\beta}_1 \times \{0\} \times \mathbb{R}$) near $+\infty$, and constant near $\partial\Sigma$.

Choose an almost complex structure J on $\Sigma \times [0, 1] \times \mathbb{R}$ agreeing with J_0 near $-\infty$, J_1 near $+\infty$, such that J is split near $\partial\Sigma$, tamed by the split symplectic form, and the fibers of both π_Σ and $\pi_{\mathbb{D}}$ are J-holomorphic.

Given an intersection point $\vec{x}^0$ in $(\Sigma, \vec{\alpha}_0, \vec{\beta}_0)$ and $\vec{x}^1$ in $(\Sigma, \vec{\alpha}_1, \vec{\beta}_1)$, the set of homology classes of curves $\pi_2(\vec{x}^0, \vec{x}^1)$ connecting $\vec{x}^0$ and $\vec{x}^1$ has an obvious meaning. Given an admissible pair Γ, and $A \in \pi_2(\vec{x}^0, \vec{x}^1)$, we let $\mathcal{M}^{A,O}$ denote the moduli space of J-holomorphic curves in $\left((\Sigma\backslash\mathfrak{z}) \times [0, 1] \times \mathbb{R}, (\vec{\alpha} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta} \times \{0\} \times \mathbb{R})\right)$ satisfying the same conditions as the corresponding object in Section 5.1. The new feature is that since J is not translation invariant, $\mathcal{M}^{A,O}$ may be generically nonempty even if $\mathrm{ind}(A, O) = 0$; the expected dimension of $\mathcal{M}^{A,O}$ is now $\mathrm{ind}(A, O)$, not

$\mathrm{ind}(A,O)-1$. We assume that J is chosen so that the $\mathcal{M}^{A,O}$ are transversely cut out; this can easily be arranged (cf. Section 4.1).

Define a map $\Phi : \mathrm{CF}(\vec{\alpha}_0,\vec{\beta}_0) \to \mathrm{CF}(\vec{\alpha}_1,\vec{\beta}_1)$ by

$$\Phi(\vec{x}^0) = \sum_{\substack{A\in\pi_2(\vec{x}^0,\vec{x}^1)\\ (O,B(\vec{x}^0))\text{admissible}\\ \mathrm{ind}(A,O)=0}} \left(\#\mathcal{M}^{A,O}\right)(O,B(\vec{x}^0))\otimes\vec{x}^1.$$

Extend Φ to a map $\Phi : \mathrm{CF}(\vec{\alpha}_0,\vec{\beta}_0) \to \mathrm{CF}(\vec{\alpha}_1,\vec{\beta}_1)$ by $\Phi(\Gamma\vec{x}^0)=\Gamma\Phi(\vec{x}^0)$.

Lemma 6.0.3 *The map Φ is a chain map.*

Proof As usual, the proof proceeds by considering the boundary of the index 1 moduli spaces. The obvious analogs of Propositions 4.2.1 and 4.2.2 imply that this boundary consists of level splittings, collapses in the order, and splitting a join curve. This implies $d\circ\Phi(\vec{x})+\Phi\circ d(\vec{x})=0$. Consequently,

$$\begin{aligned} d\circ\Phi(\Gamma\vec{x})+\Phi\circ d(\Gamma\vec{x}) &= d\left(\Gamma\cdot\Phi(\vec{x})\right)+\Phi\left((d\Gamma)\vec{x}+\Gamma\cdot\Phi(d\vec{x})\right)\\ &= (d\Gamma)\Phi(\vec{x})+\Gamma d\Phi(\vec{x})+(d\Gamma)\Phi(\vec{x})+\Gamma\Phi(d\vec{x})\\ &= 0 \end{aligned}$$

as desired. □

Similarly, choosing a family of Lagrangian cylinders C'_α (respectively C'_β) agreeing with $\vec{\alpha}_1\times\{1\}\times\mathbb{R}$ (respectively $\vec{\beta}_1\times\{0\}\times\mathbb{R}$) near $-\infty$ and $\vec{\alpha}_0\times\{1\}\times\mathbb{R}$ (respectively $\vec{\beta}_0\times\{0\}\times\mathbb{R}$) near $+\infty$, together with an appropriate almost complex structure J' on $\Sigma\times[0,1]\times\mathbb{R}$, we have a map $\Phi' : \mathrm{CF}(\vec{\alpha}_1,\vec{\beta}_1)\to\mathrm{CF}(\vec{\alpha}_0,\vec{\beta}_0)$ defined in exactly the same way as Φ. We will check that $\Phi'\circ\Phi$ is chain homotopic to the identity; the proof that $\Phi\circ\Phi'$ is chain homotopic to the identity is symmetric. This will, of course, immediately imply Proposition 6.0.2.

Let $R\gg 0$ be large; we will say exactly how large presently. Let t be the $\mathbb{R}$-coordinate on $\Sigma\times[0,1]\times\mathbb{R}$. Translate the C_α, C_β and J so that C_α (respectively C_β, J) agrees with $\vec{\alpha}_1\times\{1\}\times\mathbb{R}$ (respectively $\vec{\beta}_1\times\{0\}\times\mathbb{R}$, J_1) for $t>0$. Translate the

C'_α, C'_β and J' so that C'_α (respectively C'_β, J') agrees with $\vec{\alpha}_1 \times \{1\} \times \mathbb{R}$ (respectively $\vec{\beta}_1 \times \{0\} \times \mathbb{R}$, J_1) for $t < R$. Define

$$C^1_\alpha = \begin{cases} C_\alpha & \text{if} t \leq R \\ C'_\alpha & \text{if} t \geq R \end{cases} \quad \text{and} \quad C^1_\beta = \begin{cases} C_\beta & \text{if} t \leq R \\ C'_\beta & \text{if} t \geq R \end{cases} \quad \text{and} \quad J^1 = \begin{cases} J & \text{if} t \leq R \\ J' & \text{if} t \geq R \end{cases}.$$

Given intersection points $\vec{x}^0$ and $\vec{y}^0$ in $(\Sigma, \vec{\alpha}_0, \vec{\beta}_0)$, $A \in \pi_2(\vec{x}, \vec{y})$ and a two-level ordered list O, let $\mathcal{M}_1^{A,O}$ denote the moduli space of holomorphic curves in $\left(\Sigma \times [0,1] \times \mathbb{R}, C^1_\alpha \cup C^1_\beta\right)$ satisfying the conditions from Section 5.1 in the homology class A with asymptotics O at east ∞. For R sufficiently large, $\mathcal{M}_1^{A,O}$ is homeomorphic to

$$\bigcup_{\substack{\vec{x}^1 \text{an intersection point of} (\Sigma, \vec{\alpha}_1, \vec{\beta}_1) \\ A_1 \in \pi_2(\vec{x}^0, \vec{x}^1), A_2 \in \pi_2(\vec{x}^1, \vec{y}) \\ O_1 O_2 = O, A_1 + A_2 = A}} \mathcal{M}^{A_1,O_1} \times \mathcal{M}^{A_2,O_2}.$$

(This statement uses analogs of Propositions 4.2.1 and 4.2.2 in the case of splitting along a hypersurface.) Let $C^0_\alpha = \vec{\alpha}_0 \times \{1\} \times \mathbb{R}$, $C^0_\beta = \vec{\beta}_0 \times \{0\} \times \mathbb{R}$ and $J^0 = J_0$. Let C^t_α (respectively C^t_β), $t \in [0,1]$, be a family of Lagrangian cylinders interpolating between C^0_α and C^1_β (respectively C^0_β and C^1_β). Let J^t be a generic almost complex structure interpolating between J^0 and J^1. We assume that C^t_β and J^t are constant near east ∞. For $t \in (0,1)$, define $\mathcal{M}_t^{A,O}$ in exactly the same way as $\mathcal{M}_1^{A,O}$, with respect to C^t_α, C^t_β and J^t.

Given A and O with $\operatorname{ind}(A,O) = -1$, there is a finite list of t_i for which $\mathcal{M}_{t_i}^{A,O} \neq \emptyset$. So, it makes sense to define a map $H : \mathrm{CF}_0(\Sigma, \vec{\alpha}_0, \vec{\beta}_0, \mathfrak{z}) \to \mathrm{CF}(\Sigma, \vec{\alpha}_0, \vec{\beta}_0, \mathfrak{z})$ by

$$H(\vec{x}^0) = \sum_{\substack{A \in \pi_2(\vec{x}^0, \vec{y}^0) \\ (O, B(\vec{x}^0)) \text{admissible} \\ \operatorname{ind}(A,O) = -1}} \# \left(\bigcup_{t \in [0,1]} \mathcal{M}_t^{A,O} \right) (O, B(\vec{x}^0)) \otimes \vec{y}^0.$$

Extend H to a map $\mathrm{CF} \to \mathrm{CF}$ by setting $H(\Gamma \vec{x}^0) = \Gamma H(\vec{x}^0)$.

Lemma 6.0.4 $\Phi' \circ \Phi - \mathrm{Id} = d \circ H + H \circ d$.

Proof This follows by considering the boundary of

$$\bigcup_{\substack{A \in \pi_2(\vec{x}^0, \vec{y}^0) \\ (O, B(\vec{x}^0))\text{admissible} \\ \text{ind}(A,O)=0}} \bigcup_{t \in [0,1]} \mathcal{M}_t^{A,O}.$$

The parameter R was chosen large enough that the boundary at $t = 1$ corresponds to $\Phi' \circ \Phi$. The boundary at $t = 0$ consists entirely of trivial disks $\vec{x}^0 \times [0,1] \times \mathbb{R}$, and so corresponds to the identity map. Boundary components occurring at $t \in (0,1)$ correspond to either collapses of the order, level splitting, or splitting a "join curve" at east ∞. The sum of these phenomenon is, clearly, $d \circ H + H \circ d$. □

As we remarked earlier, Proposition 6.0.2 is now immediate.

Next we turn to stabilization. By a stabilization we mean taking the connect sum of $(\Sigma, \vec{\alpha}, \vec{\beta})$ with the standard (genus one) Heegaard diagram for S^3. A stabilization occurs in some component of $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$.

Proposition 6.0.5 *If the Heegaard diagrams* $\left(\Sigma_g, \vec{\alpha}, \vec{\beta}, \mathfrak{z}\right)$ *and*

$$\left(\Sigma_{g+1}, \vec{\alpha}' = \vec{\alpha} \cup \{\alpha_{g+1}\}, \vec{\beta}' = \beta \cup \{\beta_{g+1}\}, \mathfrak{z}'\right)$$

differ by a stabilization in the component of $\Sigma \backslash (\vec{\alpha} \cup \vec{\beta})$ *containing* $\mathfrak{z}$ *then* $\mathrm{CF}\left(\Sigma_g, \vec{\alpha}, \vec{\beta}, \mathfrak{z}\right)$ *and* $\mathrm{CF}\left(\Sigma_{g+1}, \vec{\alpha}', \vec{\beta}', \mathfrak{z}'\right)$ *are isomorphic differential* $\mathcal{A}$*-modules.*

Proof Let $x_{\text{new}} = \alpha_{g+1} \cap \beta_{g+1}$. Define a map

$$\Phi : \mathrm{CF}\left(\Sigma_g, \vec{\alpha}, \vec{\beta}, \mathfrak{z}\right) \to \mathrm{CF}\left(\Sigma_{g+1}, \vec{\alpha}', \vec{\beta}', \mathfrak{z}'\right)$$

by sending $\Gamma \otimes \vec{x}$ to $\Gamma \otimes (\vec{x} \cup \{x_{\text{new}}\})$. This is obviously an isomorphism of $\mathcal{A}$-modules. There is an obvious correspondence between $\pi_2^{\Sigma}(\vec{x}, \vec{y})$ and $\pi_2^{\Sigma'}(\vec{x} \cup \{x_{\text{new}}\}, \vec{y} \cup \{x_{\text{new}}\})$. Further, since the holomorphic curves we considered are not allowed to cover $\mathfrak{z}$, for $A \in \pi_2^{\Sigma}(\vec{x}, \vec{y}) = \pi_2^{\Sigma'}(\vec{x} \cup \{x_{\text{new}}\}, \vec{y} \cup \{x_{\text{new}}\})$, the moduli spaces $\mathcal{M}_{\Sigma}^{A,O}$ and $\mathcal{M}_{\Sigma'}^{A,O}$ are homeomorphic. □

Recall from Lemma 2.2.2 that this is all the stabilization invariance we need.

Chapter 7

Holomorphic triangles and handleslide invariance

In this section we adapt the holomorphic triangle construction of [OS04d, Section 8] and [Lip, Section 10] to prove invariance of the Heegaard Floer complex under handleslides. The proof of invariance under handleslides between the α-circles is the same as in the closed case ([OS04d, Section 9], [Lip, Section 11]), so we omit it. The proof of invariance under handleslides between β-circles or of a β-arc over a β-circle involves a few new complications; after some preliminary definitions about triangles in Section 7.1, we prove this handleslide invariance in Section 7.2.

7.1 Generalities on triangles

In this section, let Σ be a surface of genus g with a single boundary component, which we still sometimes view as a puncture. Let $\alpha_1, \cdots, \alpha_{2k_1}$ (respectively $\beta_1, \cdots, \beta_{2k_2}$, $\gamma_1, \cdots, \gamma_{2k_3}$) be a $2k_1$- (respectively $2k_2$-, $2k_3$-) tuple of arcs with boundary on $\partial\Sigma$, and $\alpha_{2k_1+1}, \cdots, \alpha_{g+k_1}$ (respectively $\beta_{2k_2+1}, \cdots, \beta_{g+k_2}$, $\gamma_{2k_3+1}, \cdots, \gamma_{g+k_3}$) be a $(g-k_1)$- (respectively $(g-k_2)$-, $(g-k_3)$-) tuple of circles in Σ. We assume that the α_i (respectively β_i, γ_i) are pairwise disjoint, that $\alpha_i \pitchfork \beta_j$, $\beta_j \pitchfork \gamma_k$, $\alpha_i \pitchfork \gamma_k$, and $\alpha_i \cap \beta_j \cap \partial\Sigma = \beta_j \cap \gamma_k \cap \partial\Sigma = \alpha_i \cap \gamma_k \cap \partial\Sigma = \alpha_i \cap \beta_j \cap \gamma_k = \emptyset$. We call such a quadruple $(\Sigma, \vec{\alpha} = \{\alpha_i\}, \vec{\beta} = \{\beta_j\}, \vec{\gamma} = \{\gamma_k\})$ a *Heegaard triple*. As before, we choose a point

$\mathfrak{z} \in \Big(\partial\Sigma\backslash(\vec{\alpha}\cup\vec{\beta}\cup\vec{\gamma})\Big)$. The quintuple $(\Sigma, \vec{\alpha}, \vec{\beta}, \vec{\gamma}, \mathfrak{z})$ is a *pointed Heegaard triple.*

There are obvious generalizations of various definitions introduced for pointed Heegaard diagrams to pointed Heegaard triples. By a *intersection point between* $\vec{\alpha}$ *and* $\vec{\beta}$ we mean a g-tuple of points $\{x_i \in \alpha_{\sigma(i)} \cap \beta_{\tau(i)}\}$ such that exactly one x_i lies on each α-circle and β-circle, and no two distinct x_i lie on the same α-arc or the same β-arc. There are obvious corresponding notions of *intersection points between* $\vec{\beta}$ *and* $\vec{\gamma}$ and *intersection points between* $\vec{\alpha}$ *and* $\vec{\gamma}$. If it is clear from context, we may not specify which circles an intersection point is between.

By a α *Reeb chord* we mean an arc in $\partial\Sigma\backslash\mathfrak{z}$ with endpoints on $\vec{\alpha}\cap\partial\Sigma$; there are corresponding notions of β *Reeb chord* and γ *Reeb chord.* We will be interested in two-level ordered lists of Reeb chords. Let O be a two-level ordered list of α Reeb chords and B a set of k_1 α-arcs. Then the definition of (O, B) being an a *admissible pair* carries over from Section 3.1; the same is true for β Reeb chords and β-arcs, and γ Reeb chords and γ-arcs.

Let T denote a disk with three punctures on the boundary. Let e_1, e_2 and e_3 denote the three arcs on ∂T, enumerated clockwise, and p_{ij} the puncture between e_i and e_j. Given intersection points $\vec{x}$ between $\vec{\alpha}$ and $\vec{\beta}$, $\vec{y}$ between $\vec{\beta}$ and $\vec{\gamma}$ and $\vec{z}$ between $\vec{\alpha}$ and $\vec{\gamma}$ let $\pi_2(\vec{x}, \vec{y}, \vec{z})$ denote the homology classes of maps

$$(S, \partial S) \to \Big(\Sigma \times T, (\vec{\alpha}\times e_1) \cup (\vec{\beta}\times e_2) \cup (\vec{\gamma}\times e_3)\Big)$$

asymptotic to $\vec{x}$ at p_{12}, $\vec{y}$ at p_{23} and $\vec{z}$ at p_{13}.

Fix almost complex structures $J_{\alpha,\beta}$, $J_{\beta,\gamma}$ and $J_{\alpha,\gamma}$ on $\Sigma \times [0,1] \times \mathbb{R}$ satisfying properties (**J1**)–(**J4**). Fix an almost complex structure J on $\Sigma \times T$ such that

(**JT1**) $\pi_t : \Sigma \times T \to T$ is J-holomorphic

(**JT2**) the fibers of $\pi_\Sigma : \Sigma \times T \to \Sigma$ are J-holomorphic

(**JT3**) J is split near the puncture p of Σ and

(**JT4**) J agrees with $J_{\alpha,\beta}$ near p_{12}, with $J_{\beta,\gamma}$ near p_{23} and with $\bar{J}_{\alpha,\gamma}$ near p_{13}.

There is an obvious generalization of the definition of *decorated sources* from Section 4.3 to the context of triangles, as well as the notion of respectful holomorphic maps $S^\diamond \to \Sigma \times T$ consistent with ordered lists (O_1, O_2, O_3). Let $A \in \pi_2(\vec{x}, \vec{y}, \vec{z})$

and $(O_1, B(\vec{x}))$, $(O_2, B(\vec{y}))$ and $(O_3, B(\vec{z}))$ be admissible pairs of α-, β- and γ-Reeb chords respectively. Let $\mathcal{M}^{A,O_1,O_2,O_3}$ denote the union over all decorated sources $S^\diamond$ consistent with (O_1, O_2, O_3) of the moduli space of smooth, embedded, finite-energy holomorphic maps $u : (S^\diamond, \partial S^\diamond) \to \left(\Sigma \times T, (\vec{\alpha} \times e_1) \cup (\vec{\beta} \times e_2) \cup (\vec{\gamma} \times e_3)\right)$ in the homology class A without components on which $\pi_T \circ u$ is constant. (Compare Section 4.3.)

Given a homology class A and ordered lists of Reeb chords O_1, O_2 and O_3, it follows from the arguments in Section 4.4 that for $(u : S \to \Sigma \times T) \in \mathcal{M}^{A,O_1,O_2,O_3}$, the Euler characteristic of S is determined by (A, O_1, O_2, O_3). It follows that there is a well-defined expected dimension $\mathrm{ind}(A, O_1, O_2, O_3)$ of $\mathcal{M}^{A,O_1,O_2,O_3}$ depending only on (A, O_1, O_2, O_3).

If any of $\vec{\alpha} \cap \partial\Sigma$, $\vec{\beta} \cap \partial\Sigma$ or $\vec{\gamma} \cap \partial\Sigma$ is empty then we will omit the corresponding (necessarily empty) list of Reeb chords from the notation. So, for instance, if $k_1 = 0$ then we write $\mathcal{M}^{A,O_2,O_3}$ to mean $\mathcal{M}^{A,\emptyset,O_2,O_3}$.

7.2 Handleslide invariance

We will focus on invariance under sliding a β-arc over a β-circle; the proof of invariance when sliding a β-circle over a β-circle is similar but slightly easier. (As noted earlier, invariance under handleslides between α-circles is essentially the same as in [OS04d] or [Lip].)

As always, fix a pointed Heegaard diagram with boundary $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$. To $\vec{\beta} = \{\beta_1, \cdots, \beta_{2k}, \beta_{2k+1}, \cdots, \beta_{g+k}\}$ we associate a lists $\vec{\beta}' = \{\beta'_1, \cdots, \beta'_{g+k}\}$ and $\vec{\beta}^H = \{\beta^H_1, \cdots, \beta^H_{g+k}\}$ of arcs and circles as follows. For $i = 1, \cdots, 2k$, let β'_i be a small perturbation of β_i, intersecting β_i transversely in one point θ'_i, disjoint from β_j for $i \neq j$, so that there are two holomorphic disks $D'_{i,\pm}$ in

$$(\Sigma \times [0, 1] \times \mathbb{R}, (\beta_i \times \{1\} \times \mathbb{R}) \cup (\beta'_i \times \{0\} \times \mathbb{R}))$$

asymptotic to $\theta'_i \times [0, 1]$ at $-\infty$ and a Reeb chord $\gamma'_{i,\pm}$ between β_i and β'_i at $+$east infinity; see Figure 7.1. For $i = k+1, \cdots, g+k$, let β'_i be a small perturbation of β_i

intersecting β_i transversely in two points. Let θ'_i be the intersection point between β_i and β'_i so that there are two holomorphic disks $D'_{i,\pm}$ in

$$(\Sigma \times [0,1] \times \mathbb{R}, (\beta_i \times \{1\} \times \mathbb{R}) \cup (\beta'_i \times \{0\} \times \mathbb{R}))$$

asymptotic to $\theta'_i \times [0,1]$ at $-\infty$; see Figure 7.1. For B' a k element subset of $\{1, \cdots, 2k\}$, let $\Theta' = \{\theta'_i | i \in B'\} \cup \{\theta'_{2k+1}, \cdots, \theta'_{g+k}\}$.

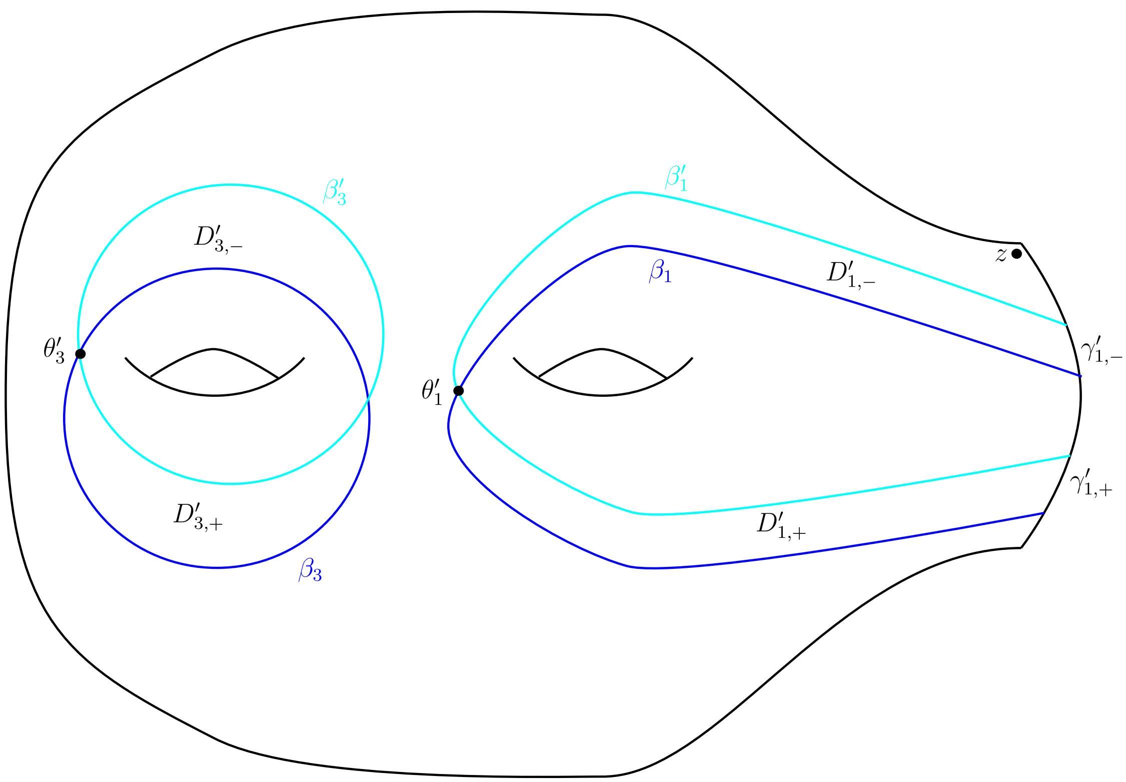


Figure 7.1: The β- and β'-arcs and circles. β_2 and β'_2 are omitted.

For $i = 2, \cdots, k$, let β_i^H be a small perturbation of β_i so that

- β_i^H and β_i intersect transversely in a single point θ_i^H.
- β_i^H and β'_i intersect transversely in a single point $\theta_i^{H\prime}$.

- there are two holomorphic disks $D^H_{i,\pm}$ in

$$\big(\Sigma \times [0,1] \times \mathbb{R}, (\beta_i \times \{1\} \times \mathbb{R}) \cup (\beta^H_i \times \{0\} \times \mathbb{R})\big)$$

asymptotic to $\theta^H_i \times [0,1]$ at $-\infty$.

- there are two holomorphic disks ${D^H_{i,\pm}}'$ in

$$\big(\Sigma \times [0,1] \times \mathbb{R}, (\beta^H_i \times \{1\} \times \mathbb{R}) \cup (\beta'_i \times \{0\} \times \mathbb{R})\big)$$

asymptotic to $\theta^{H'}_i \times [0,1]$ at $-\infty$.

See Figure 7.2. Let β^H_1 be a circle obtained by handlesliding β_1 over β_{k+1}, disjoint

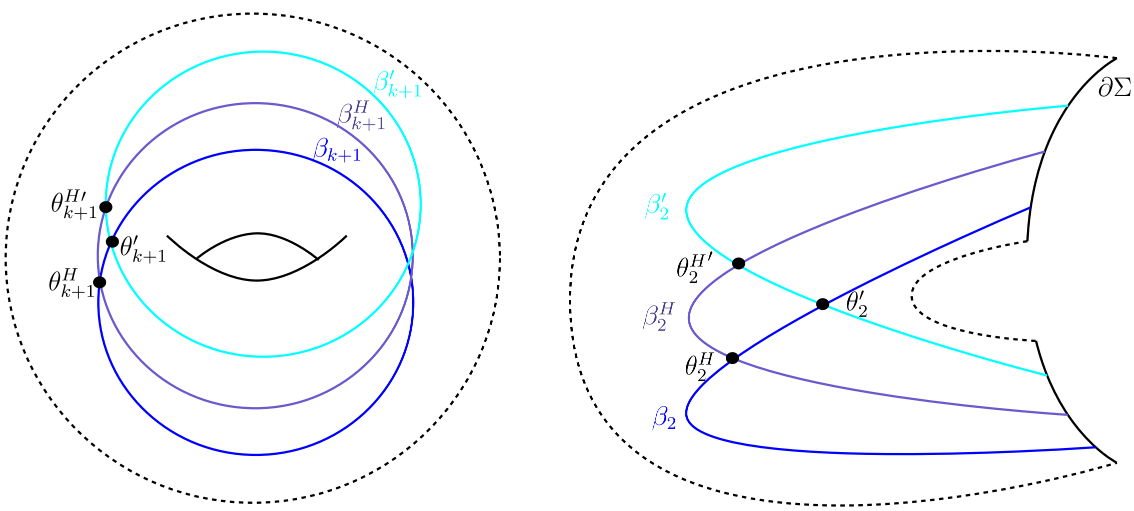


Figure 7.2: Two regions in Σ.

from the β_i and β'_i for $i \neq 1$, intersecting β_1 in a single point θ^H_1 and β'_1 in a single point $\theta^{H'}_1$, so that there is a unique holomorphic disk $D^H_{1,+}$ in

$$\Big(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\beta}_1 \times \{1\} \times \mathbb{R}) \cup (\beta^H_1 \times \{0\} \times \mathbb{R})\Big)$$

(respectively ${D^H_{1,+}}'$ in $\Big(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\beta}^H_1 \times \{1\} \times \mathbb{R}) \cup (\beta'_1 \times \{0\} \times \mathbb{R})\Big)$) asymptotic to $\theta^H_1 \times [0,1]$ (respectively $\theta^{H'}_1 \times [0,1]$) at $-\infty$ and $\gamma^H_{1,+}$ (respectively $\gamma^{H'}_{1,+}$)

at +east infinity. See Figure 7.3. For $i = 2k+1, \cdots, g+k$, let β_i^H be a small

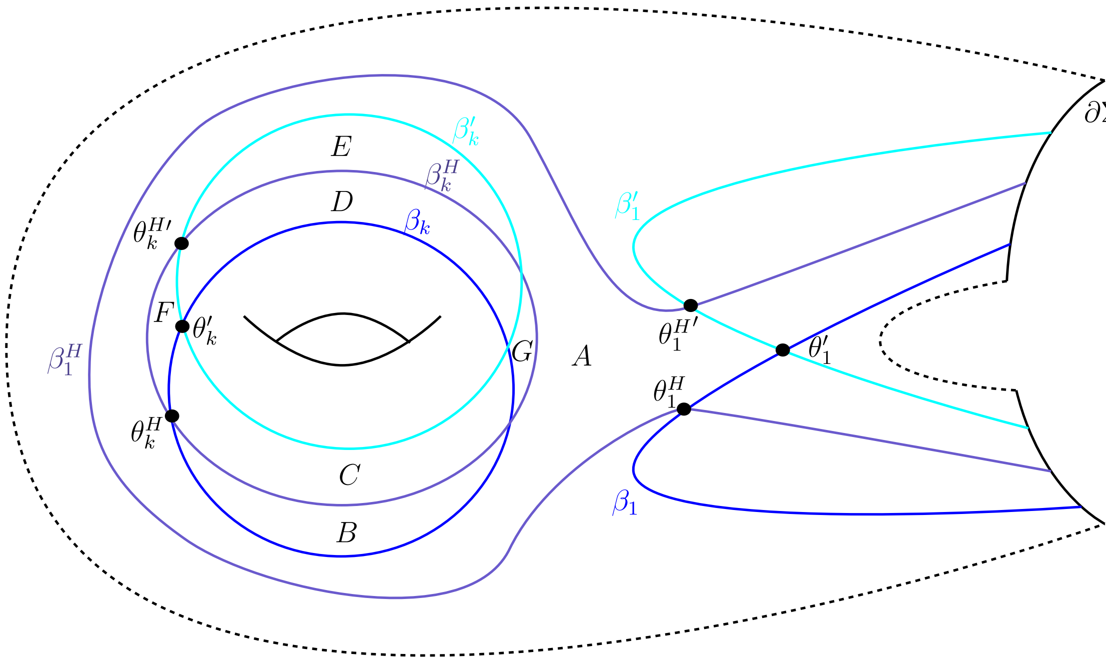


Figure 7.3: Yet another region in Σ. Capital letters label components of $\Sigma\backslash(\vec{\beta}\cup\vec{\beta}'\cup\vec{\beta}^H)$.

perturbation of β_i intersecting β_i and β_i' transversely in two points each. Let θ_i^H be the intersection point between β_i and β_i^H so that there are two holomorphic disks $D_{i,\pm}^H$ in $\big(\Sigma\times[0,1]\times\mathbb{R}, (\beta_i\times\{0\}\times\mathbb{R})\cup(\beta_i^H\times\{1\}\times\mathbb{R})\big)$. Let $\theta_i^{H\prime}$ be the intersection point between β_i^H and β_i' so that there are two holomorphic disks $D_{i,\pm}^{H\ \prime}$ in $\big(\Sigma\times[0,1]\times\mathbb{R}, (\beta_i^H\times\{0\}\times\mathbb{R})\cup(\beta_i'\times\{1\}\times\mathbb{R})\big)$. See Figure 7.2. For B^H and B k element subsets of $\{1,\cdots,2k\}$, let $\Theta_{B^H}^H = \{\theta_i^H | i\in B^H\}\cup\{\theta_{2k+1}^H,\cdots,\theta_{k+g}^H\}$ and $\Theta_B^{H\prime} = \{\theta_i^{H\prime} | i\in B\}\cup\{\theta_{2k+1}^{H\ \prime},\cdots,\theta_{k+g}^{H\ \prime}\}$.

Let $\mathrm{CF}(\alpha,\beta)$, $\mathrm{CF}(\alpha,\beta')$ and $\mathrm{CF}(\alpha,\beta^H)$ denote the Heegaard-Floer differential modules associated to $(\Sigma,\vec{\alpha},\vec{\beta},\mathfrak{z})$, $(\Sigma,\vec{\alpha},\vec{\beta}',\mathfrak{z})$ and $(\Sigma,\vec{\alpha},\vec{\beta}^H,\mathfrak{z})$ respectively.

Notice that there are obvious identifications between the algebras $\mathcal{A}$ associated to $(\partial\Sigma,\vec{\beta}\cap\partial\Sigma)$, $(\partial\Sigma,\vec{\beta}'\cap\partial\Sigma)$ and $(\partial\Sigma,\vec{\beta}^H\cap\partial\Sigma)$, and we will denote each of these three algebras by $\mathcal{A}$.

We first define a map $F_{\alpha,\beta,\beta'}: \mathrm{CF}(\alpha,\beta)\to\mathrm{CF}(\alpha,\beta')$. Given an intersection point $\vec{x} \in \mathrm{CF}_0(\alpha, \beta)$, define

$$F_{\alpha,\beta,\beta'}(\vec{x}) = \sum_{\vec{y}} \sum_{\substack{B' \subset \{1,\cdots,2k\} \\ A \in \pi_2(\vec{x}, \Theta'_{B'}, \vec{y}) \\ (O, B(\vec{x})) \in \mathcal{A}}} \sum_{\substack{O = O_1 \cdot O_2 \\ \mathrm{ind}(A, O_1, O_2) = 0}} \left[\# \left(\mathcal{M}^{A,O_1,O_2}\right)(O, B(\vec{x}))\right] \vec{y}$$

Define $F_{\alpha,\beta,\beta'}((O,B)\vec{x}) = (O,B)F_{\alpha,\beta,\beta'}(\vec{x})$.

Observe that $F_{\alpha,\beta,\beta'}$ respects the decomposition of CF into $\mathrm{Spin}^{\mathbb{C}}$-structures.

Lemma 7.2.1 *The map $F_{\alpha,\beta,\beta'}$ is a chain map, i.e., $d \circ F_{\alpha,\beta,\beta'} = F_{\alpha,\beta,\beta'} \circ d$.*

Proof In the closed case, this is a special case of [Lip, Lemma 10.18]. In our setting, the proof is more involved. As one would expect, the proof involves an analysis of the one-dimensional moduli spaces. As we will see, while these moduli spaces can degenerate in new and interesting ways, they do so an even number of times.

Suppose that $A \in \pi_2(\vec{x}, \Theta'_{B'}, \vec{y})$ and $\mathrm{ind}(A, O_1, O_2) = 1$. Then $\mathcal{M}^{A,O_1,O_2}$ potentially has four kinds of ends:

1. Degenerations at p_1 (i.e., $\vec{x}$), corresponding to $\mathcal{M}^{A',O'_1,O_2} \times \mathcal{M}^{B,O''_1}$ where $B \in \pi_2(\vec{x}, \vec{x}')$, $A' \in \pi_2(\vec{x}', \Theta', \vec{y})$, and $O_1 = O''_1 \cdot O'_1$. These correspond to $F_{\alpha,\beta,\beta'} \circ d(\vec{x})$.

2. Degenerations at p_2 (i.e., $\Theta'_{B'}$). These will require further consideration.

3. Degenerations at p_3 (i.e., $\vec{y}$), corresponding to $\mathcal{M}^{A',O_1,O'_2} \times \mathcal{M}^{B,O''_2}$ where $B \in \pi_2(\vec{y}', \vec{y})$, $A' \in \pi_2(\vec{x}, \Theta', \vec{y}')$, and $O_2 = O'_2 \cdot O''_2$. These correspond to the part of $d \circ F_{\alpha,\beta,\beta'}(\vec{x})$ coming from differentiating the generator of CF_0.

4. Degenerations at east ∞, not at p_2. These correspond to the other part of $d \circ F_{\alpha,\beta,\beta'}(\vec{x})$, coming from differentiating the coefficient in $\mathcal{A}$.

Except for 2, this would prove that $F_{\alpha,\beta,\beta'}$ is a chain map. We will show that degenerations at p_2 come in pairs. (Essentially, type 2 degenerations correspond to Reeb chords passing from Γ_1 to Γ_2.)

A degeneration at p_2 yields a curve u in

$$\left(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\beta} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}' \times \{0\} \times \mathbb{R})\right)$$

and a curve v in $\Sigma \times T$. Inspecting $(\Sigma, \vec{\beta}, \vec{\beta}')$, the only such curves u consist of disjoint unions of some of the disks $D'_{i,\pm}$ and some trivial disks. For simplicity, assume for the moment that v consists of a single nontrivial component. (Generically, this will be the case if O has no nontrivial microscopic partitions. To prove this requires the remark on gluing a few paragraphs below.) Without loss of generality, assume the nontrivial component is $D'_{i,+}$.

If the nontrivial component $D'_{i,+}$ is contained in the interior of Σ (i.e., $i > k$), then there is also a family degenerating to $(D'_{i,-}, v)$, so such degenerations come in pairs.

If the nontrivial component $D'_{i,+}$ touches east infinity (i.e., $i \leq k$) then in u there is Reeb chord γ between $\vec{\beta}$ and $\vec{\beta}'$ mapped by v to p_2 and east ∞. Either $\beta_{\text{down}}(\gamma) = \beta_i$ or $\beta_{\text{up}}(\gamma) = \beta'_i$. In the former case, let $\beta_{\text{up}}(\gamma) = \beta'_j$. Then u can be glued to one of $D'_{j,\pm}$ to produce a family of curves converging to the chain $(D'_{j,-}, u)$. (See the remark on gluing a few paragraphs below.) (This has the effect of passing a Reeb chord from O_2 to O_1.) In the latter case, let $\beta_{\text{down}}(\gamma) = \beta_k$. Then u can be glued to one of $D'_{k,\pm}$ to obtain a family of holomorphic curves converging to $(D'_{k,\pm}, u)$. (This has the effect of passing a Reeb chord from O_1 to O_2.) It follows that type 2 degenerations at a single Reeb chord occur in pairs, as desired. See Figure 7.4 for an example illustrating this phenomenon.

If there is more than one nontrivial component, the preceding discussion applies, but with a microscopic partition of Reeb chords rather than a single Reeb chord γ.

Remark on gluing. The argument appears to require a new gluing lemma, but in fact does not: one can fill-in the puncture of Σ so that the β_i meet the β'_i transversely at p. The result then follows from the obvious non-cylindrical analog of Proposition 4.2.2. $\square$

Lemma 7.2.2 *The map $F_{\alpha,\beta,\beta'}$ is an isomorphism of differential $\mathcal{A}$-modules.*

Proof The analogous statement in the closed case is [OS04d, Proposition 9.8] or [Lip, Proposition 11.4]. The proofs of these two results given in the closed case are somewhat different; we will adapt the former to our setting. The idea, which goes back at least to Floer ([Flo95, Lemma 3.6]), is that the map $F_{\alpha,\beta,\beta'}$ preserves the

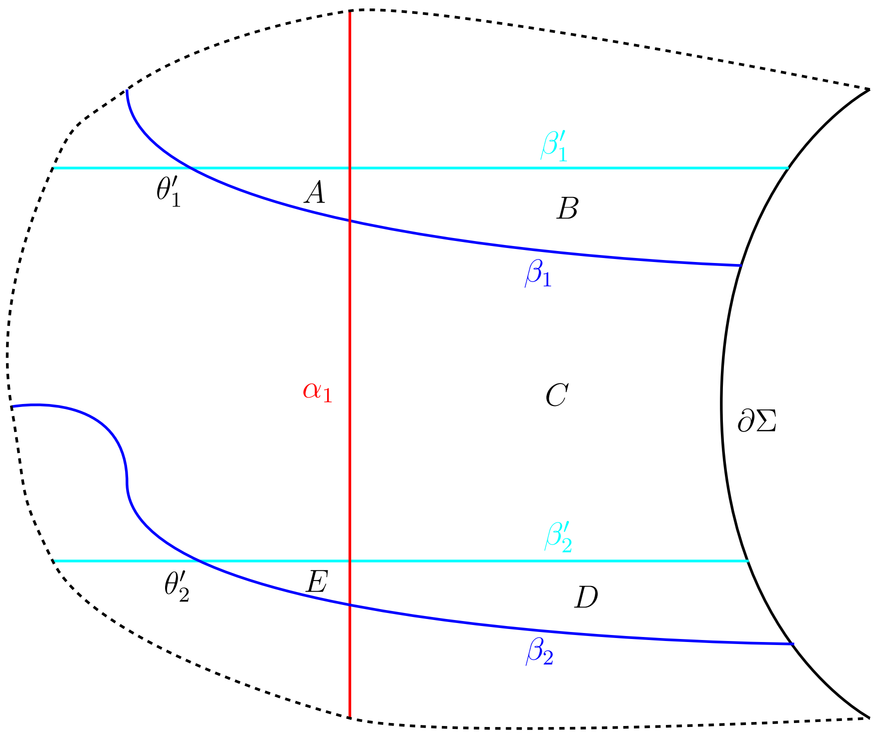


Figure 7.4: Part of a Heegaard diagram illustrating a pair of degenerations. The curves with domain $A+B+C$ degenerate to $(A+B, C)$. The triangle with domain $C+D+E$ degenerate to $(D+E, C)$. Observe that $A+B = D'_{1,\pm}$ and $D+E = D'_{2,\pm}$.

ω-energy filtration, and the lowest filtered part is an isomorphism; this is enough to imply $F_{\alpha,\beta,\beta'}$ is a homotopy equivalence. (This is analogous to the statement that an upper triangular matrix with nonzero entries on the diagonal is invertible.) Formally, we will use the following algebraic lemma, which is [OS04d, Lemma 9.10], the proof of which is left (by both them and us) to the reader.

Lemma 7.2.3 *Let $F : A \to B$ be a map of filtered groups, which is decomposed as a sum $F = F_0 + \ell$, where F_0 is a filtration-preserving isomorphism and ℓ has (strictly) lower order than F_0. Suppose that the filtration on B is bounded below. Then F is an isomorphism of groups.*

Remark. If A and B have additional structure, i.e., belong to some category $\mathcal{C}$ with a forgetful map to the category of groups, and isomorphisms in $\mathcal{C}$ are bijective morphisms, then the lemma obviously implies that if F is a morphism in $\mathcal{C}$ then F is an isomorphism in $\mathcal{C}$. This applies, in particular, if $\mathcal{C}$ is the category of differential $\mathcal{A}$-modules.

Turning to the proof proper, choose an area form on Σ as in Proposition 2.7.2 so that every domain in $\pi_2^{\alpha,\beta}(\vec{x},\vec{x})$ has zero signed area. Further arrange that $\mathrm{Area}(D_i^+) = \mathrm{Area}(D_i^-)$ for $i = 1,\cdots,g+k$; this is easy to do. It follows that every domain in $\pi_2^{\alpha,\beta'}(\vec{x},\vec{x})$ has zero signed area.

Now, for each $\mathrm{Spin}^{\mathbb{C}}$-structure $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(Y)$ we define a filtration $\mathcal{F}$ on $\mathrm{CF}^{\mathfrak{s}}(\alpha,\beta)$. We will define $\mathcal{F}$ by giving a map $\mathcal{F} : \mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta) \to \mathbb{R}$ with discrete (in fact, finite) image, extending $\mathcal{F}$ to a map $\mathcal{F} : \mathrm{CF}^{\mathfrak{s}}(\alpha,\beta) \to \mathbb{R}$ by $\mathcal{F}(\Gamma\vec{x}) = \mathcal{F}(\vec{x})$, and then inducing a filtration on $\mathrm{CF}^{\mathfrak{s}}(\alpha,\beta)$ from the filtration $\mathbb{R} = \cup_{t\in\mathbb{R}}\{s \in \mathbb{R} | s < t\}$ of $\mathbb{R}$.

Choose a generator $\vec{x}$ of $\mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta)$, and define $\mathcal{F}(\vec{x}) = 0$. For $\vec{y}$ any other generator of $\mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta)$ let $A \in \pi_2^{\alpha,\beta}(\vec{x},\vec{y})$ and define $\mathcal{F}(\vec{y}) = -\mathrm{Area}(A)$. Since the area of any element of $\pi_2(\vec{x},\vec{x})$ is zero, the definition of $\mathcal{F}(\vec{y})$ is independent of the choice of A. If the coefficient of $\vec{z}$ in $d\vec{y}$ is nonzero then there is some $A \in \pi_2(\vec{y},\vec{z})$ with non-negative coefficients, and hence positive area. This implies that $\mathcal{F}$ does in fact induce a filtration. Extend $\mathcal{F}$ to $\mathrm{CF}^{\mathfrak{s}}(\alpha,\beta)$ by $\mathcal{F}(\Gamma\vec{x}) = \mathcal{F}(\vec{x})$.

Define a filtration $\mathcal{F}_0'$ on $\mathrm{CF}^{\mathfrak{s}}(\alpha,\beta')$ in exactly the same way as $\mathcal{F}$ on $\mathrm{CF}^{\mathfrak{s}}(\vec{\alpha},\vec{\beta})$, using the same area form and some intersection point $\vec{x}'$ in $\mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta')$. Let $A_{\vec{x},\vec{x}'} \in \pi_2(\vec{x},\vec{x}',\Theta')$ and let $\mathcal{F}'$ be the filtration $\mathcal{F}$ shifted by $\mathrm{Area}(A_{\vec{x},\vec{x}'})$. That is, $\mathcal{F}'(\vec{y}') = \mathcal{F}_0'(\vec{y}') + \mathrm{Area}(A_{\vec{x},\vec{x}'})$. Since elements of $\pi_2(\vec{x},\vec{x})$ and $\pi_2(\vec{x}',\vec{x}')$ have zero signed area, the definition of $\mathcal{F}'$ is independent of the choice of $A_{\vec{x},\vec{x}'}$.

We check that $F_{\alpha,\beta,\beta'}$ respects the filtrations $\mathcal{F}$ and $\mathcal{F}'$. Fix intersection points $\vec{y} \in \mathrm{CF}^{\mathfrak{s}}(\alpha,\beta)$ and $\vec{y}' \in \mathrm{CF}^{\mathfrak{s}}(\alpha,\beta')$ and $B \in \pi_2(\vec{x},\vec{y})$. Then $\mathcal{F}(\vec{y}) = -\mathrm{Area}(B)$. For any positive $A \in \pi_2(\vec{y},\vec{y}',\Theta')$, $A + B - A_{\vec{x},\vec{x}'} \in \pi_2(\vec{x}',\vec{y}')$ and $\mathrm{Area}(A + B - A_{\vec{x},\vec{x}'}) > \mathrm{Area}(B) - \mathrm{Area}(A_{\vec{x},\vec{x}'})$. It follows that if $\vec{y}'$ occurs with nonzero coefficient in $F_{\alpha,\beta,\beta'}(\vec{y})$ then $\mathcal{F}_0'(\vec{y}') < \mathcal{F}(\vec{y}) + \mathrm{Area}(A_{\vec{x},\vec{x}'})$ so $\mathcal{F}'(\vec{y}') < \mathcal{F}(\vec{y})$ as desired.

Since there are only finitely many intersection points, both $\mathcal{F}$ and $\mathcal{F}'$ are finite, and so in particular bounded below.

For convenience, choose a Riemannian metric on Σ inducing the chosen area form. If the β- and β'-curves are sufficiently close with respect to this metric then for every point $x \in \alpha_i \cap \beta_j$ there is a closest point $x' \in \alpha_i \cap \beta'_j$ and a small embedded triangle T_x in Σ with $\partial T_x \subset (\alpha_i \cup \beta_j \cup \beta'_j$ and corners at x, x', and θ'_j. (The triangle T_x is characterized by these properties and being entirely contained in the isotopy region between β_j and β'_j.) See Figure 7.4, in which regions A and E are both examples of such triangles. If the β- and β'-curves are close enough then each T_x has smaller area than any region in $\Sigma\backslash(\vec{\alpha} \cup \vec{\beta})$ or $\Sigma\backslash(\vec{\alpha} \cup \vec{\beta}')$.

Given an intersection point $\vec{x} = \{x_i\} \in \mathrm{CF}_0^{\mathfrak{s}}(\alpha, \beta)$ let $\vec{x}' = \{x'_i\}$, the "closest intersection point in $\mathrm{CF}_0^{\mathfrak{s}}(\alpha, \beta')$ to $\vec{x}$". It follows that given $\vec{x} = \{x_i\}$, the domain $\sum_i T_{x_i} \in \pi_2(\vec{x}, \vec{x}')$ has smallest area among positive domain in $\pi_2(\vec{x}, \vec{y}')$ (for all $\vec{y}'$). So, $F_{\alpha,\beta,\beta'} = F_0 + \ell$ where $F_0(\{x_i\}) = \{x'_i\}$ and ℓ has strictly lower order than F_0. Obviously $\mathrm{CF}^{\mathfrak{s}}(\vec{\alpha}, \vec{\beta}, \mathfrak{z}) \to \mathrm{CF}^{\mathfrak{s}}(\vec{\alpha}, \vec{\beta}', \mathfrak{z})$ is an isomorphism of filtered groups. Thus Lemma 7.2.3 implies the result. □

Next, we define and study a map $F_{\alpha,\beta,\beta^H} : \mathrm{CF}(\alpha, \beta) \to \mathrm{CF}(\alpha, \beta^H)$. Given an intersection point $\vec{x} \in \mathrm{CF}_0(\alpha, \beta)$, define

$$F_{\alpha,\beta,\beta^H}(\vec{x}) = \sum_{\vec{y}} \sum_{\substack{B^H \subset \{1,\cdots,2k\} \\ A \in \pi_2(\vec{x}, \Theta^H_{B^H}, \vec{y}) \\ (O, B(\vec{x})) \in \mathcal{A}}} \sum_{\substack{O = O_1 \cdot O_2 \\ \mathrm{ind}(A, O_1, O_2) = 0}} \left[\# \left(\mathcal{M}^{A,O_1,O_2}\right)(O, B(\vec{x}))\right] \vec{y}$$

(This formula should look familiar: it is the same formula we used to define $F_{\alpha,\beta,\beta'}(\vec{x})$ with $\Theta^H_{B^H}$ in place of $\Theta'_{B'}$.) Define $F_{\alpha,\beta,\beta^H}((O, B)\vec{x}) = (O, B)F_{\alpha,\beta,\beta^H}(\vec{x})$. Observe that again F_{α,β,β^H} respects the decomposition of CF into $\mathrm{Spin}^{\mathbb{C}}$-structures.

Lemma 7.2.4 *F_{α,β,β^H} is a chain map.*

Proof The proof is the same as the proof of Lemma 7.2.1, except that the analysis of the curves occurring in $\left(\Sigma \times [0, 1] \times \mathbb{R}, (\vec{\beta} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}^H \times \{0\} \times \mathbb{R})\right)$ is somewhat more complicated than for $\left(\Sigma \times [0, 1] \times \mathbb{R}, (\vec{\beta} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}' \times \{0\} \times \mathbb{R})\right)$. Fortunately, this analysis was already carried out in the closed case, so we need only quote relevant results.

Notice that in $(\Sigma, \vec{\beta}, \vec{\beta}')$ there is an annular region A_1 from θ^H_1 with boundary on β_1, β_2 and β^H_1 asymptotic to $\gamma^H_{1,-}$ at east infinity and a second annular region A_2

from θ_1^H with boundary on β_1, β_1^H and β_2^H asymptotic to $\gamma_{1,-}^H$ at east infinity.

We must check that the homology classes of $D_{i,\pm}^H$ each admit (algebraically) one representative in $\left(\Sigma \times [0,1] \times \mathbb{R}, (\vec{\beta} \times \{1\} \times \mathbb{R}) \cup (\vec{\beta}^H \times \{0\} \times \mathbb{R})\right)$, and one of the regions A_1 or A_2 admits one representative (algebraically) and the other zero representatives (algebraically). The statement about the $D_{i,\pm}^H$ is clear. The statement about A_1 and A_2 follows from the proof of [OS04d, Lemma 9.4]. The rest of the proof is the same as the proof of Lemma 7.2.1. □

There is also a map $F_{\alpha,\beta^H,\beta'} : \mathrm{CF}(\alpha, \beta^H) \to \mathrm{CF}(\alpha, \beta')$ defined in the same way as F_{α,β,β^H}.

Lemma 7.2.5 *The composition $F_{\alpha,\beta^H,\beta'} \circ F_{\alpha,\beta,\beta^H}$ is chain homotopic to $F_{\alpha,\beta,\beta'}$.*

Proof The proof involves considering maps to $\Sigma \times R$ where R is a rectangle. The idea, which is the same as in [OS04d] or [Lip], is the following. The moduli space of conformal structures on a rectangle is an interval, whose endpoints correspond to the two possible degenerations of a rectangle into two triangles glued at one point. Up to chain homotopy, the map defined by counting holomorphic curves in $\Sigma \times R$ is independent of the choice of conformal structures on R. It follows that the two compositions of triangle maps corresponding to the degenerations of R are chain homotopic. One composition is obviously $F_{\alpha,\beta^H,\beta'} \circ F_{\alpha,\beta,\beta^H}$ while it is not hard to check the other is $F_{\alpha,\beta,\beta'}$. More details follow.

The argument decomposes naturally into two parts: generalities on holomorphic rectangles and a detailed study of holomorphic curves in

$$\left(\Sigma \times T, (\vec{\beta} \times e_1 \cup \vec{\beta}^H \times e_2 \cup \vec{\beta}' \times e_3)\right).$$

We discuss the second first.

We need to understand index 0 holomorphic curves

$$u : (S, \partial S) \to \left(\Sigma \times T, (\vec{\beta} \times e_1 \cup \vec{\beta}^H \times e_2 \cup \vec{\beta}' \times e_3)\right)$$

asymptotic to $\Theta_{B^H}^H$ and $\Theta_B^{H'}$ at p_{12} and p_{23} respectively. Inspecting the diagram, if $B^H \neq B$ then no such curve exists. Further, for $j \neq j'$, $j, j' \notin \{1, k\}$, the punctures mapped by u to θ_j^H and $\theta_j^{H'}$ must lie on different connected components of S. It follows again from inspection that for $j \notin \{1, k\}$ the component asymptotic to θ_j^H must be the small triangles with corners at θ_j^H, $\theta_j^{H'}$ and $\theta_j{}'$ (see Figure 7.2; this is slightly more obvious for $j > k$ than for $1 < j < k$). By the Riemann mapping theorem, each of these regions has a unique holomorphic representative in $\Sigma \times T$ with respect to any split almost complex structure.

The components asymptotic to θ_1^H, $\theta_1^{H'}$ θ_k^H and $\theta_k^{H'}$ require a more complicated analysis. Again, inspecting the diagram there are three annular domains which might support index zero holomorphic curves: the domains $A + D + E + 2F + G$, $A+B+E+F$, and $A+B+C+2F+G$ in Figure 7.3. We will show by a somewhat indirect argument that, for a generic perturbation of the β-, β'- and β^H-curves (or generic complex structure J on $\Sigma \times T$), the total number of holomorphic curves among these three domains is (algebraically) one. (Presumably this also follows directly from a sufficiently clever conformal mapping argument.)

One can choose another Heegaard triple-diagram with boundary $(\Sigma_0, \vec{\beta}_0, \vec{\beta}_0^H, \vec{\beta}_0')$ so that Σ and Σ_0 glue to a closed Heegaard diagram $(\Sigma \natural \Sigma_0, \vec{\beta} \natural \vec{\beta}_0, \vec{\beta}^H \natural \vec{\beta}_0^H, \vec{\beta}' \natural \vec{\beta}_0')$ which corresponds to a single handleslide in the standard Heegaard diagram $(\Sigma \natural \Sigma_0, \vec{\beta} \natural \vec{\beta}_0, \vec{\beta}' \natural \vec{\beta}_0')$ for $\#(2g + 2k) S^1 \times S^2$. Let $\Theta'_{B'} \natural \Theta'_0$ (respectively $\Theta^H_{B^H} \natural \Theta^H_0$, $\Theta^{H'}_B \natural \Theta^{H'}_0$) denote the top-dimensional generator of $\widehat{\mathrm{CF}}(\Sigma \natural \Sigma_0, \vec{\beta} \natural \vec{\beta}_0, \vec{\beta}' \natural \vec{\beta}_0') = \widehat{\mathrm{CF}}(\#(2g + 2k) S^1 \times S^2)$ (respectively $\widehat{\mathrm{CF}}(\Sigma \natural \Sigma_0, \vec{\beta} \natural \vec{\beta}_0, \vec{\beta}^H \natural \vec{\beta}_0^H)$, $\widehat{\mathrm{CF}}(\Sigma \natural \Sigma_0, \vec{\beta}^H \natural \vec{\beta}_0^H, \vec{\beta}' \natural \vec{\beta}_0')$). It is proved in [OS04d, Lemma 9.7] that $F_{\beta \natural \beta_0, \beta^H \natural \beta_0^H, \beta' \natural \beta_0'}(\Theta^H_{B^H} \natural \Theta^H_0, \Theta^{H'}_B \natural \Theta^{H'}_0) = \Theta'_{B'} \natural \Theta'_0$. (They prove this for a slightly different Heegaard diagram, but the result for this Heegaard diagram follows since the differentials vanish and the triangle maps in homology are invariants of the bordism specified by the Heegaard triple diagram.) It follows that algebraically exactly one of the three annular domains under consideration must support a holomorphic curve.

In summary, there is algebraically a single holomorphic curve u with index 0 connecting $\Theta'_{B'}$, $\Theta^H_{B^H}$ and $\Theta^{H'}_B$. The image of $\pi_\Sigma \circ u$ is contained in a compact subregion of Σ. There are no other index 0 holomorphic curves in

$$\left(\Sigma \times T, (\vec{\beta} \times e_1 \cup \vec{\beta}^H \times e_2 \cup \vec{\beta}' \times e_3)\right)$$

asymptotic to $\Theta^H_{B^H}$ at p_{12} and $\Theta^{H'}_B$ at p_{23}.

Now that we understand holomorphic curves in $(\Sigma, \vec{\beta}, \vec{\beta}^H, \vec{\beta}')$ we return to the main argument. Let R denote a topological rectangle (disk with four boundary punctures), with edges $f_1, \cdots, f_4$ (enumerated clockwise). For consecutive edges f_i and f_j, let w_{ij} be the vertex between f_i and f_j. Let $a \in (0,1)$ denote a parameter for the moduli space of conformal structures on R so that as $a \to 0$ (R, j_a) degenerates along an arc connecting f_1 and f_3 and as $a \to 1$ (R, j_a) degenerates along an arc connecting f_2 and f_4. See Figure 7.5.

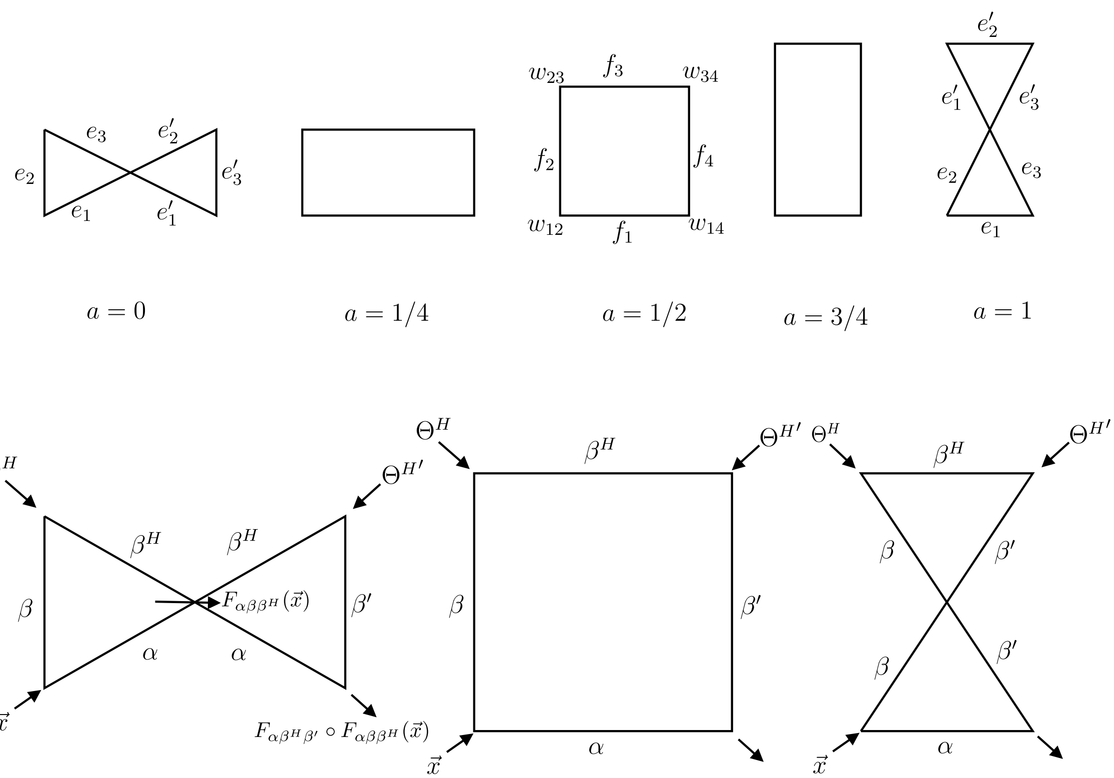


Figure 7.5: The conformal structures on a rectangle R, and boundary conditions for maps to $\Sigma \times R$.

Fix almost complex structures $J_{\alpha\beta\beta^H}$, $J_{\alpha\beta\beta'}$, $J_{\alpha\beta^H\beta'}$ and $J_{\beta\beta^H\beta'}$ satisfying (**JT1**)–(**JT4**) and achieving transversality for index 0 holomorphic curves in $\Sigma \times T$ with the obvious corresponding boundary conditions.

Fix a family $\{J_a\}_{a\in\mathbb{R}}$ of almost complex structures on $\Sigma \times R$ so that for each a, J_a satisfies the obvious analogs of Properties (**JT1**)–(**JT4**), with respect to the almost complex structure j_a on R. Further assume that

- As $a \to 0$ the family J_a degenerates to $J_{\alpha\beta\beta^H}$ on $(\Sigma\times T, \vec{\alpha}\times e_1\cup\vec{\beta}\times e_2\cup\vec{\beta}^H\times e_3)$ and $J_{\alpha\beta^H\beta'}$ on $(\Sigma\times T, \vec{\alpha}\times e_1'\cup\vec{\beta}^H\times e_2'\cup\vec{\beta}'\times e_3')$. Here, e_1 and e_1' each corresponds to "half of" f_1; e_2 to f_2; e_3 and e_2' to half of f_3 each; and e_3' to f_4.
- As $a \to 1$ the family J_a degenerates to $J_{\alpha\beta\beta'}$ on $(\Sigma\times T, \vec{\alpha}\times e_1\cup\vec{\beta}\times e_2\cup\vec{\beta}'\times e_3)$ and $J_{\beta\beta^H\beta'}$ on $(\Sigma \times T, \vec{\beta}\times e_1'\cup\vec{\beta}^H\times e_2'\cup\vec{\beta}'\times e_3')$. Here, e_1 corresponds to f_1; e_2 and e_1' to f_2; e_2' to f_3; and e_3 and e_3' to f_4.

See Figure 7.5.

Given intersection points $\vec{x} \in \mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta)$, $\vec{y} \in \mathrm{CF}_0^{\mathfrak{s}}(\alpha,\beta')$, and $B^H, B \subset \{1,\cdots,2k\}$ let $\pi_2(\vec{x},\Theta^H_{B^H},\Theta^{H'}_B,\vec{y})$ denote the homology classes of maps to $(\Sigma\times R, \vec{\alpha}\times e_1\cup\cdots)$ asymptotic to $\vec{x}$ at w_{12}, $\Theta^H_{B^H}$ at w_{23}, $\Theta^{H'}_B$ at w_{34} and $\vec{y}$ at w_{14}.

Given a homology class $A \in \pi_2(\vec{x},\Theta^H_{B^H},\Theta^{H'}_B,\vec{y})$ and two-level ordered lists of Reeb chords O_1, O_2 and O_3 let $\mathcal{M}_a^{A,O_1,O_2,O_3}$ denote the moduli space of embedded, finite-energy J_a-holomorphic curves in the homology class A from sources with decorations consistent with (O_1,O_2,O_3), for which no components have $\pi_{\mathbb{D}}\circ u$ constant. (Compare Section 4.3.) Let $\mathcal{M}^{A,O_1,O_2,O_3} = \bigcup_{a\in[0,1]}\mathcal{M}_a^{A,O_1,O_2,O_3}$.

By the index $\mathrm{ind}(u)$ of a map $u : (S,\partial S) \to (\Sigma\times T, \vec{\alpha}\times f_1\cup\cdots)$ we mean the expected dimension of the component of $\mathcal{M}_a^{A,O_1,O_2,O_3}$ containing u. By essentially the same argument as given in Section 4.4, $\mathrm{ind}(u)$ depends only on the homology class A and the asymptotics (O_1,O_2,O_3) at east infinity, and so for $A\in\pi_2(\vec{x},\Theta^H_{B^H},\Theta^{H'}_B,\vec{y})$ and (O_1,O_2,O_3) ordered lists of Reeb chords we may write $\mathrm{ind}(A,O_1,O_2,O_3)$ to denote this expected dimension.

Choose the family J_a so that it achieves transversality, as a family, for index ≤ 0 holomorphic curves in $(\Sigma\times R, \vec{\alpha}\times f_1\cup\vec{\beta}\times f_2\cup\vec{\beta}^H\times f_3\cup\vec{\beta}'\times f_4)$. Define a map

$H : \mathrm{CF}_0(\alpha, \beta) \to \mathrm{CF}_0(\alpha, \beta')$ by

$$H(\vec{x}) = \sum_{\vec{y}} \sum_{\substack{B^H, B \subset \{1, \cdots, 2k\} \\ A \in \pi_2(\vec{x}, \Theta^H_{B^H}, \Theta^{H'}_B, \vec{y}) \\ (O, B(\vec{x})) \in \mathcal{A}}} \sum_{\substack{O = O_1 \cdot O_2 \cdot O_3 \\ \mathrm{ind}(A, O_1, O_2, O_3) = -1}} \left[\# \left(\mathcal{M}^{A, O_1, O_2, O_3}\right)(O, B(\vec{x}))\right] \vec{y}.$$

If $(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ was admissible then this expression makes sense. Extend H to a map $H : \mathrm{CF}(\alpha, \beta) \to \mathrm{CF}(\alpha, \beta')$ by setting $H(\Gamma\vec{x}) = \Gamma H(\vec{x})$.

We will show that

$$F_{\alpha, \beta^H, \beta'} \circ F_{\alpha, \beta, \beta^H} + F_{\alpha, \beta, \beta'} = d \circ H + H \circ d.$$

This follows by considering the boundary of

$$\bigcup_{\substack{B^H, B \subset \{1, \cdots, 2k\} \\ A \in \pi_2(\vec{x}, \Theta^H_{B^H}, \Theta^{H'}_B, \vec{y}) \\ (O, B(\vec{x})) \in \mathcal{A}}} \bigcup_{\substack{O = O_1 \cdot O_2 \cdot O_3 \\ \mathrm{ind}(A, O_1, O_2, O_3) = 0}} \mathcal{M}^{A, O_1, O_2, O_3}.$$

This space has several kinds of ends, corresponding to different degenerations of the holomorphic curves:

- Ends corresponding to degenerations into two-level curves at w_{12}. These correspond to $H \circ d$.
- Ends corresponding to degenerations into two-level curves at w_{14}. These correspond to part of $d \circ H(\vec{x})$.
- Ends corresponding to degenerations into two-level curves at w_{23} or w_{34}. These cancel in pairs, as in the proof of Lemma 7.2.1.
- Ends corresponding to degenerations at east ∞, not at any w_{ij}. These correspond to the rest of $d \circ H(\vec{x})$.
- Ends corresponding to $\{0, 1\} \subset [0, 1]$. These correspond to $F_{\alpha, \beta^H, \beta'} \circ F_{\alpha, \beta, \beta^H}(\vec{x})$ and $F_{\alpha, \beta, \beta'}(\vec{x})$ respectively.

This proves that $F_{\alpha,\beta^H,\beta'} \circ F_{\alpha,\beta,\beta^H}$ is chain homotopic to $F_{\alpha,\beta,\beta'}$. □

Observe that if the β- and β'-curves are sufficiently close then $\mathrm{CF}(\alpha,\beta)$ and $\mathrm{CF}(\alpha,\beta')$ are exactly the same differential $\mathcal{A}$-module.

Proposition 7.2.6 *If the β- and β'-curves are sufficiently close then the maps $F_{\alpha,\beta^H,\beta'}$ and F_{α,β,β^H} are homotopy inverses to each other, and in particular,* $\mathrm{CF}(\alpha,\beta)$ *and* $\mathrm{CF}(\alpha,\beta^H)$ *are chain homotopy equivalent differential $\mathcal{A}$-modules.*

Proof From Lemma 7.2.5 we know that $F_{\alpha,\beta^H,\beta'} \circ F_{\alpha,\beta,\beta^H}$ is chain homotopic to $F_{\alpha,\beta,\beta'}$, which by Lemma 7.2.2 is an isomorphism of differential $\mathcal{A}$-modules.

We could define a fourth set of curves $\beta^{H'}$, isotopic to the β^H, and intersecting the β^H- and β'-curves in the same way the β'-curves intersect the β- and β^H-curves. We would then have an isomorphism $F_{\alpha,\beta^H,\beta^{H'}} : \mathrm{CF}(\alpha,\beta^H) \to \mathrm{CF}(\alpha,\beta^{H'})$ and chain maps $F_{\alpha,\beta^H,\beta'} : \mathrm{CF}(\alpha,\beta^H) \to \mathrm{CF}(\alpha,\beta')$ and $F_{\alpha,\beta',\beta^{H'}} : \mathrm{CF}(\alpha,\beta') \to \mathrm{CF}(\alpha,\beta^{H'})$ so that $F_{\alpha,\beta',\beta^{H'}} \circ F_{\alpha,\beta^H,\beta'}$ is chain homotopic to $F_{\alpha,\beta^H,\beta^{H'}}$, which is also an isomorphism of differential $\mathcal{A}$-modules.

If the β- and β'-curves are sufficiently close then $\mathrm{CF}(\alpha,\beta)$ and $\mathrm{CF}(\alpha,\beta')$ are exactly the same differential $\mathcal{A}$-module. Similarly, if the β^H- and $\beta^{H'}$-curves are sufficiently close then $\mathrm{CF}(\alpha,\beta^H)$ and $\mathrm{CF}(\alpha,\beta^{H'})$ are exactly the same differential $\mathcal{A}$-modules. Further, the maps $F_{\alpha,\beta',\beta^{H'}}$ and F_{α,β,β^H} are exactly the same. It follows that F_{α,β,β^H} and $F_{\alpha,\beta^H,\beta'}$ are homotopy inverses to each other, proving the result. □

The fact that, up to chain homotopy, CF is independent of the choice of Heegaard diagram follows immediately from Lemma 2.2.2 and Propositions 6.0.2, 6.0.5 and 7.2.6.

Chapter 8

Topics for further study

8.1 The gluing conjecture

The main reason to introduce invariants of bordered 3-manifolds is to be able to study the Heegaard-Floer homology of closed manifolds by cutting and gluing. We conjecture that the invariant introduced in this paper is strong enough to compute $\widehat{\mathrm{HF}}$ of a glued manifold. We make this conjecture precise presently. Fix, for now, a closed, oriented surface S and a Morse function $f : S \to \mathbb{R}$ (with a single index 2 and a single index 0 critical point). Let $\mathcal{A}$ be the algebra associated to $f : S \to \mathbb{R}$ in chapter 3, and let $\overline{\mathcal{A}}$ denote the algebra associated to $f : (-S) \to \mathbb{R}$, where $-S$ denotes S with the opposite orientation. Let $\mathcal{A} - \mathrm{Mod}$ denote the category of finitely generated, free differential $\mathcal{A}$-modules, and $\mathbb{F}_2 - \mathrm{Mod}$ the category of differential $\mathbb{F}_2$-modules.

Conjecture 8.1.1 *There is a covariant functor* $G : (\mathcal{A} - \mathrm{Mod}) \times (\overline{\mathcal{A}} - \mathrm{Mod}) \to \mathbb{F}_2 - \mathrm{Mod}$ *with the following properties:*

1. *The functor G is exact in each factor.*
2. *The functor G descends to a map of homotopy categories.*
3. *Given $\partial Y_1^3 = S$ and $\partial Y_2^3 = -S$, $\widehat{\mathrm{CF}}(Y_1 \cup_\partial Y_2)$ is chain homotopy equivalent to $G\,(\mathrm{CF}(Y_1), \mathrm{CF}(Y_2))$.*

In the rest of this section, we describe a program for proving the gluing conjecture. We start by precisely describing how to define the $\mathbb{F}_2$-vector space $G(M_1, M_2)$. Let I be the ideal of $\mathcal{A}$ generated by all (O, B) with $O \neq \emptyset$, so $\mathcal{A}_0 = \mathcal{A}/I$. This makes $\mathcal{A}_0$ into an $\mathcal{A}$-algebra. Observe that there is an obvious isomorphism $\mathcal{A}_0 \cong \overline{\mathcal{A}}_0$; identify $\overline{\mathcal{A}}_0$ with $\mathcal{A}$ via this isomorphism.. Define a pairing $P : \mathcal{A}_0 \otimes_{\mathbb{F}_2} \mathcal{A}_0 \to \mathbb{F}_2$ by

$$P(\,(\emptyset, B_1), (\emptyset, B_2)) = \begin{cases} 1 & \text{if } B_1 \cap B_2 = \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

This makes $\mathbb{F}_2$ into a $(\mathcal{A}_0 \otimes_{\mathbb{F}_2} \mathcal{A}_0)$-algebra.

Now, define

$$G(M_1, M_2) = \mathbb{F}_2 \otimes_{\mathcal{A}_0 \otimes_{\mathbb{F}_2} \mathcal{A}_0} \left((\mathcal{A}_0 \otimes_{\mathcal{A}} M_1) \otimes_{\mathbb{F}_2} (\mathcal{A}_0 \otimes_{\mathcal{A}} M_2) \right).$$

If $M_1 = \mathrm{CF}(\Sigma_1, \vec{\alpha}_1, \vec{\beta}_1)$ and $M_2 = \mathrm{CF}(\Sigma_2, \vec{\alpha}_2, \vec{\beta}_2)$ it is easy to see that $G(M_1, M_2) = \widehat{\mathrm{CF}}(\Sigma_1 \cup_\partial \Sigma_2, \vec{\alpha}_1 \cup_\partial \vec{\alpha}_2, \vec{\beta}_1 \cup_\partial \vec{\beta}_2)$as a $\mathbb{F}_2$-vector space.

Next we discuss how one might define a differential on $G(M_1, M_2)$. Given Heegaard diagrams with boundary $(\Sigma_1, \vec{\alpha}_1, \vec{\beta}_1, \mathfrak{z}_1)$ and $(\Sigma_2, \vec{\alpha}_2, \vec{\beta}_2, \mathfrak{z}_2)$ with

$$\partial(\Sigma_1, \vec{\alpha}_1, \vec{\beta}_1, \mathfrak{z}_1) = -\partial(\Sigma_2, \vec{\alpha}_2, \vec{\beta}_2, \mathfrak{z}_2),$$

let $\Sigma = \Sigma_1 \cup_\partial \Sigma_2$, $\vec{\alpha} = \vec{\alpha}_1 \cup \vec{\alpha}_2$, $\vec{\beta} = \vec{\beta}_1 \cup_\partial \vec{\beta}_2$ and $\mathfrak{z} = \mathfrak{z}_1 = \mathfrak{z}_2$. Suppose that $\vec{x} = \vec{x}^1 \cup \vec{x}^2$ and $\vec{y} = \vec{y}^1 \cup \vec{y}^2$. Given $A_1 \in \pi_2(\vec{x}^1, \vec{y}^1)$ and an ordered list of Reeb chords O_1 there is an evaluation map

$$\mathrm{ev} : \mathcal{M}^{A_1, O_1} \to \overline{\mathbb{R}^{|O|}/\mathbb{R}}$$

given by recording the $\mathbb{R}$-coordinates of the various Reeb chords at east ∞. Here, $\mathbb{R}^{|O|}/\mathbb{R}$ is the quotient with respect to the diagonal action by addition, and $\overline{\mathbb{R}^{|O|}/\mathbb{R}}$ is a compactification this quotient obtained by allowing the distance between Reeb chords to go to zero or infinity. (Topologically, $\overline{\mathbb{R}^{|O|}/\mathbb{R}}$ is a cube of dimension $|O| - \varepsilon(O)$.) It is not hard to show that for $A \in \pi_2(\vec{x}, \vec{y})$, and an appropriate choice of complex structure on $\Sigma \times [0,1] \times \mathbb{R}$,

$$\mathcal{M}^A = \bigcup_{\substack{A = A_1 +_\partial A_2 \\ O \text{compatible with} A_1}} \mathcal{M}^{A_1,O} \times_{\mathrm{ev}} \mathcal{M}^{A_2,\bar{O}}. \tag{8.1}$$

Here, $\overline{O}$ corresponds to the map $\mathcal{A} \to \overline{\mathcal{A}}$ given by $(O,B) \mapsto (-O, B^{|O|})$, where $-O$ is obtained from O by completely reversing the order. (This decomposition of $\mathcal{M}^A$ requires a compactness and a gluing theorem, but both can be deduced easily from standard results.)

Observe that even though A may have index 1, the (A_1, O) and (A_2, O) into which A decomposes may have high index. On the other hand, the invariant $\mathrm{CF}(Y_i, \partial Y_i)$ takes into account only the index 1 moduli spaces. Therefore, to prove the gluing conjecture, one wants to reconstruct the higher index moduli spaces from the index 1 moduli spaces.

The $\mathcal{M}^{A_1,O}$ satisfy the relation

$$\partial\mathcal{M}^{A_1,O} = \sum_{\substack{A_1 = A_1' + A_1'' \\ O = O' \cdot O''}} \mathcal{M}^{A_1',O'} \times \mathcal{M}^{A_1'',O''} + \sum_{O = F(O')} F(\mathcal{M}^{A,O'}) + \sum_{O \in \mathrm{decol}(O')} \mathcal{M}^{A,O'}. \tag{8.2}$$

Here, F corresponds to maps induced by gluing-in curves at east ∞; the simplest instance of such is gluing-in a join-curve; in this case, F is essentially just the operation join.

Via the evaluation map ev, $\mathcal{M}^{A_1,O}$ represents some chain in $\overline{\mathbb{R}^{|O|}/\mathbb{R}}$; by formula (8.1) this chain is all we care about. Since $\overline{\mathbb{R}^{|O|}/\mathbb{R}}$ is contractible, up to homology, this chain is determined by its boundary. By formula (8.2), the boundary is determined by simpler chains. Inductively, one can see that up to homology, all chains are determined by (counts of) the zero-dimensional moduli spaces.[1]

[1]The attentive reader may object that homologous chains have the same boundary. What we really mean is some kind of iterated homology, deforming first the zero-dimensional corners, then the one-dimensional corners, and so on up. These homologies can be chosen to enjoy certain coherence properties for different moduli spaces. In particular, they can be chosen so that formula (8.2) remains true.

It remains to check two things.

1. Changing the moduli spaces by (coherent) homology changes the glued chain complex by chain homotopy equivalence.

2. Changing the differential $\mathcal{A}$-module CF by chain homotopy equivalence changes the glued chain complex by chain homotopy equivalence.

If both of these hold, one would have proved at least parts (2) and (3) of the gluing conjecture. The proof of at least the first point is likely to proceed along relatively standard lines.

Note that if this description of gluing is correct, it is explicit enough to be computed combinatorially. Recall that $\overline{\mathbb{R}^{|O|}/\mathbb{R}}$ is a cube. One could deform the $\mathcal{M}^{A,O}$ so that their images are affine linear subspaces of this cube with specified corners. The gluing problem, then, reduces to linear arithmetic.

8.2 The relationship with knot Floer homology

This section assumes familiarity with knot Floer homology; see [OS04b] otherwise.

In Section 2.4 we showed how to obtained a Heegaard diagram with boundary $(\Sigma, \vec{\alpha} = \{\alpha_1, \cdots, \alpha_g\}, \vec{\beta} = \{\beta_1, \cdots, \beta_{g-1}, \lambda, \mu\})$ for the complement of a knot $K \hookrightarrow S^3$. Fix such a Heegaard diagram. Choose the basepoint $\mathfrak{z} \in \partial\Sigma$ so that γ_1 runs from λ to μ. Fix also a point $w \in \gamma_2 \subset \partial\Sigma$. Let $\overline{\Sigma}$ denote Σ with the puncture filled-in; λ and μ specify circles in $\overline{\Sigma}$. The points $\mathfrak{z}$ and w specify connected components of $\overline{\Sigma}\setminus(\vec{\alpha}\cup\vec{\beta})$; we will sometimes view $\mathfrak{z}$ and w as points in these connected components. Let $\gamma_4 = \gamma_1 \uplus \gamma_2$, $\gamma_5 = \gamma_2 \uplus \gamma_3$ and $\gamma_6 = \gamma_1 \uplus \gamma_2 \uplus \gamma_3$.

Let $S^3_0(K)$ denote zero surgery along K. Let $\vec{\beta}_\lambda = \{\beta_1, \cdots, \beta_{g-1}, \lambda\}$ and $\vec{\beta}_\mu = \{\beta_1, \cdots, \beta_{g-1}, \mu\}$. Observe that $(\Sigma, \vec{\alpha}, \vec{\beta}_\mu, \mathfrak{z}, w)$ is a doubly pointed Heegaard diagram for (S^3, K) and $(\Sigma, \vec{\alpha}, \vec{\beta}_\lambda, \mathfrak{z}, w)$ is a doubly pointed Heegaard diagram for $(S^3_0(K), K)$.

Let $\mathrm{CF} = \mathrm{CF}(\Sigma, \vec{\alpha}, \vec{\beta}, \mathfrak{z})$ denote the Heegaard-Floer differential module associated to $S^3 \setminus K$. As discussed in Section 8.1, there is an algebra map $\mathcal{A} \to \mathcal{A}_0 = \mathbb{F}_2 \oplus \mathbb{F}_2$. The differential in $\mathrm{CF} \otimes_{\mathcal{A}} \mathcal{A}_0$ counts only provincial curves, i.e., curves with multiplicity zero near the puncture. It is easy to see, therefore, that the homology of $\mathrm{CF} \otimes_{\mathcal{A}} \mathcal{A}_0$ is $\widehat{\mathrm{HFK}}(S^3, K) \oplus \widehat{\mathrm{HFK}}(S^3_0(K), K)$.

More generally, the entire filtered chain complex $\widehat{\mathrm{CFK}}(S^3, K)$ is contained in CF. For $\vec{x}, \vec{y} \in \widehat{\mathrm{CFK}}(S^3, K) \subset \mathrm{CF}$, terms of the form $(\gamma_5 < \gamma_5 < \cdots < \gamma_5, \mu)\vec{y}$ in $d\vec{x}$ correspond to the filtered differential on $\widehat{\mathrm{CFK}}(S^3, K)$ of $d\vec{x}$. However, it does not seem possible to extract just these terms in a homotopy-invariant way.

Let $\Theta_{\mu,\lambda}$ (respectively $\Theta_{\lambda,\mu}$, $\Theta_{\lambda,\lambda}$, $\Theta_{\mu,\mu}$) denote a top-dimensional generator of $\widehat{\mathrm{CF}}(\bar{\Sigma}, \vec{\beta}_\mu, \vec{\beta}_\lambda, \mathfrak{z}) = \mathbb{F}_2^{g-1}$ (respectively $\widehat{\mathrm{CF}}(\bar{\Sigma}, \vec{\beta}_\lambda, \vec{\beta}_\mu, \mathfrak{z}) = \mathbb{F}_2^{g-1}$, $\widehat{\mathrm{CF}}(\bar{\Sigma}, \vec{\beta}_\lambda, \vec{\beta}_\lambda, \mathfrak{z}) = \mathbb{F}_2^{g}$, $\widehat{\mathrm{CF}}(\bar{\Sigma}, \vec{\beta}_\mu, \vec{\beta}_\mu, \mathfrak{z}) = \mathbb{F}_2^{g}$). For $\vec{x} \in \widehat{\mathrm{CFK}}(S^3, K)$, terms of the form $(\gamma_2, \mu)\vec{y}$ in $d\vec{x}$ correspond to the triangle map $F(\vec{x} \otimes \Theta_{\mu,\lambda})$ induced by the Heegaard triple-diagram $(\bar{\Sigma}, \vec{\alpha}, \vec{\beta}_\mu, \vec{\beta}_\lambda, \mathfrak{z}, w)$. For $\vec{x} \in \widehat{\mathrm{CFK}}(S^3_0(K), K)$, terms of the form $(\gamma_1, \lambda)\vec{y}$ or $(\gamma_3, \lambda)\vec{y}$ in $d\vec{x}$ correspond to the triangle map $F(\vec{x} \otimes \Theta_{\lambda,\mu})$ induced by the Heegaard triple-diagram $(\bar{\Sigma}, \vec{\alpha}, \vec{\beta}_\lambda, \vec{\beta}_\mu, \mathfrak{z}, w)$. To determine which triangles correspond to (γ_1, λ) and which to (γ_3, λ) would require an appropriate twisted coefficient system. Higher order terms in $d\vec{x}$ should correspond to considering n-gons, for $n > 3$, with one puncture mapped to $\vec{x}$, the next puncture mapped to $\vec{y}$, and all other punctures mapped to $\Theta_{*,*}$. In this case, in addition to the issue of twisted coefficients, some technical discussion of appropriate perturbations of the β-circles is necessary. It would be nice to have these details completely understood.

It would be interesting to know whether our invariant of $S^3 \setminus K$ is completely determined by $\widehat{\mathrm{CFK}}(S^3, K)$. In light of the discussion in this section together with the arguments in [OSc], this seems plausible, but not obviously true. There are, fortunately or unfortunately, a dearth of small knots with isomorphic Heegaard-Floer invariants, so there is no computational evidence one way or the other.

8.3 Future directions

Assuming the gluing conjecture is correct, or perhaps even if it is incorrect, there are a number of potential applications and generalizations of the invariant developed in this thesis. We mention some of them here.

8.3.1 Other homologies, disconnected boundary

It would be nice to generalize our invariant to include analogs of HF^+, HF^- and HF^∞. While the geometric picture is clear — one would still study curves in $\Sigma \times [0,1] \times \mathbb{R}$ with certain asymptotics — how to algebrize that picture is less obvious. In this picture, there would be infinitely many Reeb chords (as one would allow them to cover $\mathfrak{z}$), as well as closed Reeb orbits. Reeb orbits could split into Reeb chords, and Reeb chords could coalesce into Reeb orbits. The algebra should make allowances for these phenomena.

In another direction, this theory should generalize to Heegaard diagrams with more than one boundary component. If one places basepoints $\mathfrak{z}_i$ on all of the boundary components, the generalization is obvious. However, for some applications one might want to have many boundary components but a single basepoint; in this case, difficulties similar to those for HF^+, HF^- and HF^∞ are likely to arise.

8.3.2 Contact structures

It is likely that one can imitate the construction in [OS05a] to define an invariant of contact structures on manifolds with boundary; it seems reasonable to expect that this invariant would take the form of an element $c(\xi) \in \mathrm{CF}(Y)$ associated to a contact structure ξ on Y, and that this invariant would glue to give the invariant of a closed manifold. This might allow one to relate the Heegaard-Floer contact invariant with the technique of cutting contact manifolds along convex surfaces.

One conjecture one might hope to be able to prove using a relative contact invariant is the following, due to P. Lisca and A. Stipsicz ([LS, Conjecture 1.4]):

Conjecture 8.3.1 *Suppose that the closed contact* 3*-manifold* (Y, ξ) *admits a contact embedding of*

$$\left(T^2 \times [0,1], \ker\left(\cos(2\pi\,\mathrm{nz})\,\mathrm{dx} - \sin(2\pi\,\mathrm{nz})\,\mathrm{dy}\right)\right).$$

(Here, x *and* y *are coordinates on* T^2 *and* z *on* $[0,1]$*.) Then the Heegaard-Floer contact invariant* $c(Y, \xi)$ *vanishes.*

As noted by Lisca and Stipsicz, this conjecture has as a corollary the following conjecture of Eliashberg, recently proved by D. Gay using completely different methods in [Gay]:

Theorem 8.3.2 *(Gay) If the closed contact* 3*-manifold* (Y, ξ) *admits a contact embedding of*

$$\big(T^2 \times [0,1], \ker\left(\cos(2\pi\, \mathrm{nz})\, \mathrm{dx} - \sin(2\pi\, \mathrm{nz})\, \mathrm{dy}\right)\big)$$

then (Y, ξ) *is not strongly fillable.*

8.3.3 Knots

As mentioned at various points earlier, in [OS04b] and [Ras03], Ozsváth and Szabó and, independently, Rasmussen, used Heegaard-Floer homology to define a knot invariant. Briefly, their invariant is defined as follows. Fix a Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta})$ for S^3 coming from some Morse function f and metric g on Y. Fix also points $z, w \in \Sigma\backslash(\vec{\alpha} \cup \vec{\beta})$. The point z (respectively w) lies on a flow line ℓ_z (respectively ℓ_w) between the index 0 and 3 critical points. Then $\ell_z \cup \ell_w$ is a knot K in S^3; any knot can be obtained this way for an appropriate choice of Heegaard diagram. Recall that $\widehat{\mathrm{HF}}(S^3)$ is defined by counting curves in $(\Sigma\backslash\{\mathfrak{z}\}) \times [0,1] \times \mathbb{R}$. The Heegaard-Floer knot invariant is defined by counting curves in $(\Sigma\backslash\{z, w\}) \times [0,1] \times \mathbb{R}$. More generally, replacing f by a Morse function with k index 0 and 3 critical points and choosing points $z_1, \cdots, z_k$ and $w_1, \cdots, w_k$ in Σ one can obtains an invariant of links with k components; see [OSa].

It should be possible to combine their construction with ours in at least two ways. The first was suggested to me by Hedden. Choose a Morse function on $\mathbb{D}^2 \times S^1$ with one index 0 (respectively 3) critical point on $(\partial\mathbb{D}^2) \times S^1$ and a second index 0 (respectively 3) critical point in the interior of $\mathbb{D}^2 \times S^1$. Associated is a (generalized) Heegaard diagram with boundary $(\Sigma, \vec{\alpha}, \vec{\beta})$. Let z be a point in $\partial\Sigma\backslash\vec{\alpha}$, and z_1, w_1 points in $\mathrm{int}(\Sigma)\backslash(\vec{\alpha} \cup \vec{\beta})$. For appropriate choices of z_1 and w_1, these points specify a knot P in $\mathbb{D}^2 \times S^1$. (P stands for "pattern.")

Suppose we are interested in the knot Floer homology of a satellite knot, with pattern P and companion K. (K stands for "kompanion.") Assuming the gluing conjecture, this homology should be completely determined by $\mathrm{CF}(S^3 \setminus K)$ and $\mathrm{CFK}(\mathbb{D}^2 \times S^1, P)$, where the second differential module denotes the obvious modification of CF taking into account the new basepoints z_1 and w_1. This result should be particularly useful for the many patterns in $\mathbb{D}^2 \times S^1$ admitting genus 1 Heegaard diagrams, for which computations can presumably be carried-out relatively easily.

The second way to combine the two ideas seeks to produce invariants of tangles. Choose a Morse function on $S^2 \times [0,1]$ with $k_i + 1$ index 0 (respectively 3) critical points on the boundary component $S^2 \times \{i\}$, and k index 0 (respectively 3) critical points in the interior of $S^2 \times [0,1]$. This gives a generalized Heegaard diagram $(\Sigma, \vec{\alpha}, \vec{\beta})$ with two boundaries $\partial_1\Sigma$ and $\partial_2\Sigma$. Fix a point $\mathfrak{z}_i$ on $\partial_i\Sigma$. Choosing $k_1 + k_2 + 2k$ basepoints w_i appropriately in Σ then specifies a tangle in $S^2 \times [0,1]$ with $k_1 + k_2$ arcs and k circles.

Imitating the construction of chapter 3, one can associate to $\partial_i\Sigma$ a differential algebra $\mathcal{A}_i$. Imitating Section 5.1, one can associate an $(\mathcal{A}_0, \mathcal{A}_1)$-bimodule CF to $(\Sigma, \vec{\alpha}, \vec{\beta})$. Assuming the (appropriate) gluing conjecture holds in this setting as well, one can then view CF as a functor from $\mathcal{A}_0$-modules to $\mathcal{A}_1$-modules. (This would be analogous to the functor-valued invariant of tangles associated in [Kho02] to Khovanov homology.) In particular, the invariant of a knot would be determined by the bimodules associated to elementary tangles. If the proof of the gluing construction is sufficiently explicit, this would presumably lead to a combinatorial algorithm for computing knot Floer homology.

One could apply similar ideas to studying the Heegaard-Floer homology of the branched double cover of a link. Let $D(L)$ denote the double cover of S^3 branched over L. In [OS05c], Ozsváth and Szabó show that $\widehat{\mathrm{HF}}(D(L))$ is an invariant of L with remarkably similar properties to the Khovanov homology $\mathrm{Kh}(L)$. Using these similarities, they construct a spectral sequence from $\mathrm{Kh}(L)$ to $\widehat{\mathrm{HF}}(D(L))$. Using Heegaard diagrams with two boundary components, it should be possible to extend this invariant to a (functor-valued) invariant of tangles. It would be interesting to know, then, how the spectral sequence would generalize to this context. Again, assuming an explicit gluing theorem, this should make combinatorial computation of $\widehat{\mathrm{HF}}(D(L))$ possible.

8.3.4 Other potential applications

Several other applications of the invariant of bordered manifolds suggest themselves. Heegaard-Floer homology detects the genus of a knot or link ([OS04a], [Nic]) and, in fact, the Thurston norm on $H_2(Y)$ for Y closed or Y a link complement ([OSe], [Nib]). It would be interesting to know, therefore, what the bordered invariant has to say about the minimal genus problem in 3-manifolds with boundary.

Another conceivable application, suggested to me by I. Agol, is to try to use the bordered invariant to bound the Heegaard genus of a manifold. In [Gar98], S. Garoufalidis shows that any unitary $(2+1)$-dimensional TQFT gives a Heegaard genus bound for 3-manifolds. It seems plausible that by "categorifying" his arguments one could use our bordered invariant to obtain a, presumably stronger, Heegaard genus bound.

Finally, Heegaard-Floer invariants of sutured 3-manifolds have recently been of substantial interest ([Juh], [Ni06], [Ghi]), leading ultimately to a proof of the remarkable fact that knot Floer homology detects fibered knots ([Nia]). Although perhaps unlikely to lead to new topological applications, it would be nice to know how these results relate to the invariants introduced in this thesis.

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