Knot Floer Homology and the Genera of Torus Knots

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1 Introduction

The goal of this paper is to provide a new computation of the genus of a torus knot. This paper will use a recent definition of the Alexander polynomial as the determinant of the winding matrix of a grid diagram. A direct proof of the invariance of this determinant will be given and then it will be shown that this determinant is the Alexander polynomial. The value in these proofs will be that they are direct and easy to understand. To obtain these goals we first discuss Knot Floer Homology.

Knot Floer Homology $\widehat{HFK}$ is an invariant for knots which is related to Heegaard Floer Homology, an invariant for three-manifolds. $\widehat{HFK}$ is a finite dimensional bi-graded vector space over $\mathbb{Z}/2$. Related to this is $\widetilde{HFK}$, which is associated to a toroidal grid diagram of a knot and dependent on the knot K and the arc index n of the diagram. The relation, more concretely from [MOS06] is the isomorphism $\widetilde{HFK} = \widehat{HFK} \bigotimes V^{n-1}$ where V is a two dimensional $\mathbb{Z}/2$ vector space. So from $\widetilde{HFK}$ it is possible to recover $\widehat{HFK}$, a knot invariant. $\widetilde{HFK}$ is related to the Alexander polynomial $\Delta_K(T)$ by the formula:

$$(1 - t^{-1})^{n-1}\Delta_K(T) = \sum_{i,j}(-1)^j t^i \dim \widetilde{HFK}_{i,j} \qquad (1)$$

A planar grid diagram is an $n \times n$ grid where every row contains exactly one X and one O, every column contains exactly one X and one O and no cell contains more than one X or O. A planar grid diagram specifies a knot or link projection as follows: draw a line between any X or O that are in the same column or row. The convention followed will be that at every crossing the vertical line will pass over the horizontal line. A toroidal grid diagram is a planar grid diagram where the top and bottom edges are identified and the left and right edges are identified. Here, $\widetilde{HFK}$ will actually be associated with a toroidal grid diagram.

In fact $\widetilde{HFK}$ is the homology of the chain complex $\widetilde{CFK}$. The generators of $\widetilde{CFK}$ are given by the n-tuples of intersection points between horizontal and vertical arcs (viewed as circles on the torus) on the diagram, with the added condition that every intersection point appears on a horizontal and vertical circle and each of the horizontal and vertical circles contains one of the intersection points. Thus there is a bijection between elements $x \in S_n$ and the generators of $\widetilde{CFK}$. This bijection can be thought of as a labelling of the elements.

Finally, we discuss the gradings on $\widetilde{CFK}$. For A,B two collections of points on the plane one defines the number $I(A, B)$. $I(A, B)$ counts the number of pairs $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$ with $a_1 < b_1$ and $a_2 < b_2$. As well, one defines $J(A, B)$ as the average of $I(A, B)$ and $I(A, B)$.

The functions $A(x) : X \to \mathbb{Z}$ and $M(x) : X \to \mathbb{Z}$ are defined by:

$$A(x) = J(x, X) - J(x, O) - \frac{1}{2}J(X, X) + \frac{1}{2}J(O, O) - \frac{n-1}{2} \tag{2}$$

$$M(x) = J(x, x) - 2J(x, O) + J(O, O) + 1 \tag{3}$$

Here $x \in \widetilde{CFK}_{ij}$ if and only if $A(x) = i$ and $M(x) = j$.

The differential is given by counting empty rectangles in the grid diagram - but for our purposes it will not be needed.

I am thankful to Robert Lipshitz for his supervision and insight. I also thank Andrew Freimuth and Yael Degany; I worked closely with them during the summer and specifically the figures in section 5 are Yael Degany's.

2 The Minesweeper Determinant

It is a theorem of Dynnikov that any two grid diagrams of the same knot can be related by a sequence of the following three moves:

1. Cyclic Permutation (Figure 9)
2. Stabilization (Figure 4)
3. Commutation (Figure 2)

For a $n \times n$ grid diagram define a matrix $M(G)_{ij} = t^{a(i,j)}$ where $a(i, j)$ is the winding number of the point (i, j). Put in another way, the minesweeper determinant of a grid diagram is arrived at by simple means: for every point on the grid diagram we associate the variable t raised to the winding number

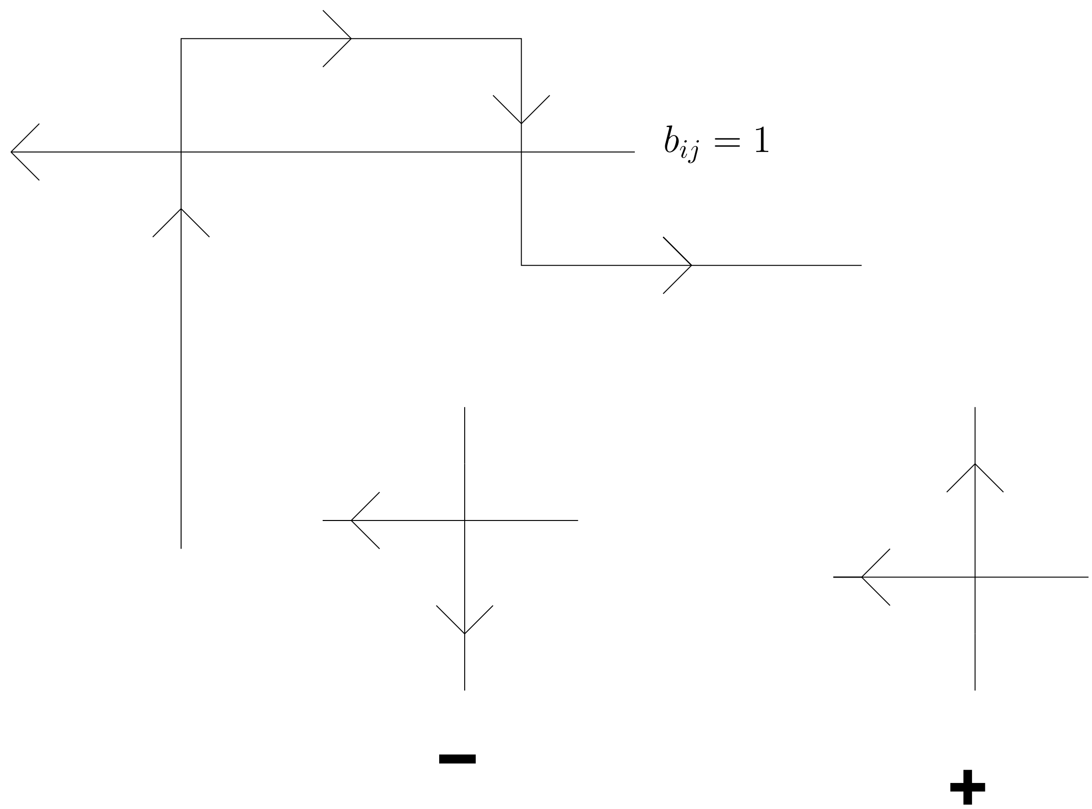


Figure 1: Entry b_{ij} is computed in a winding matrix

at that point. The purpose of the following section is to show that the minesweeper determinant of the grid diagram is unchanged up to factors of $(1-t)$, $\pm t^{\pm n}$.

A note for computing winding numbers: when finding the winding number of a point (i,j) of the winding matrix, one can draw a ray starting at that point and count the intersections of that ray with the knot projection, taking into account orientation.

Lemma 1. *Cyclic Permutation changes the minesweeper determinant by* $\pm t^{\pm n}$

Proof. Assume without loss of generality that the cyclic permutation makes the last column of a grid diagram the first column, shifting the other columns to the right.

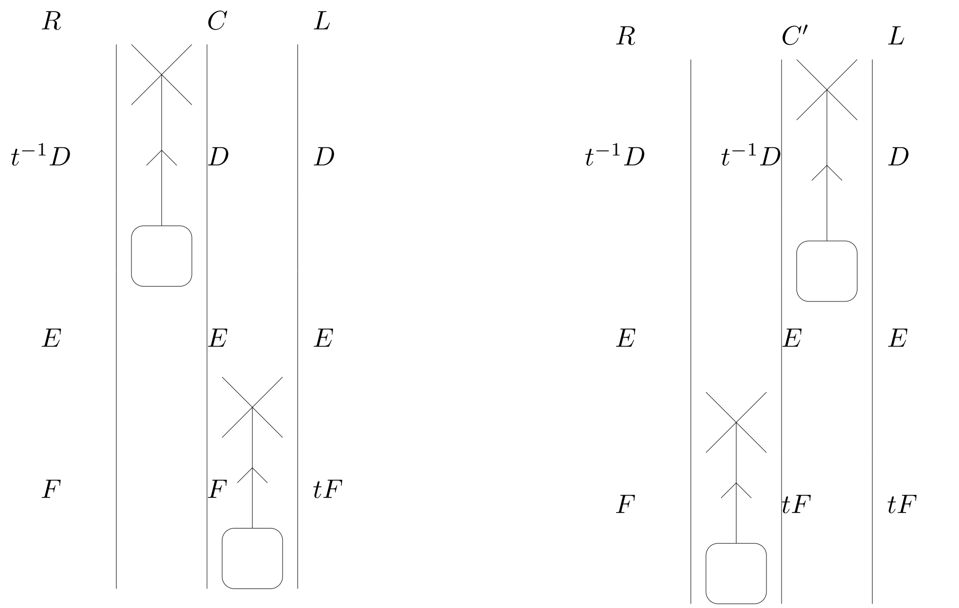


Figure 2: Commutation

Let A be the winding matrix of a grid diagram, and let B be the winding matrix after cyclic permutation. From A, let's construct an intermediate matrix B' by taking the first $n-1$ columns of A and shifting them over one to the right. More to the point, for an entry a_{ij} in A such that $j \leq n-1$, $b'_{ij} = a_{i(j+1)}$. This gives the last $n-1$ columns of B'. Fill in the first column by adding the last column of B, giving B' fully. The determinant of B and B' is the same up to sign. Finally for the k rows in between the X and O in the first column of the grid diagram for B multiply by $t^{\pm 1}$, the sign determined by orientation. This operation gives B, which shows that the cyclic permutation changes the determinant by factors of $\pm t^{\pm n}$.

$\square$

For example, let B be the winding matrix of the 5×5 diagram of the trefoil pictured in Figure 3 so that

$$B = \begin{pmatrix} 1 & t & t & t & 1 \\ 1 & t & t^2 & t^2 & t \\ 1 & 1 & t & t^2 & t \\ 1 & 1 & 1 & t & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

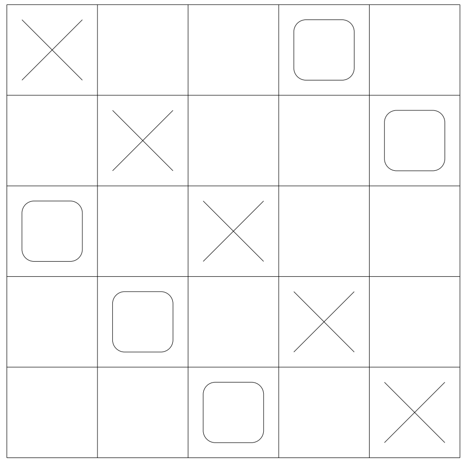


Figure 3: Grid Diagram of a Trefoil

So

$$B' = \begin{pmatrix} 1 & 1 & t & t & t \\ t & 1 & t & t^2 & t^2 \\ t & 1 & 1 & t & t^2 \\ t & 1 & 1 & 1 & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The picture above dictates that one multiplies rows $2-4$ by t^{-1}, so that one obtains the desired matrix for the new diagram of the trefoil:

$$\begin{pmatrix} 1 & 1 & t & t & t \\ 1 & t^{-1} & 1 & t & t \\ 1 & t^{-1} & t^{-1} & 1 & t \\ 1 & t^{-1} & t^{-1} & t^{-1} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Lemma 2. *Stabilization changes the determinant by $t^{\pm n}$, factors of $\pm(1-t)$.*

Proof. This Dynnikov move adds a row and column to the matrix. Let M be the original matrix and let M' be the new matrix after stabilization. To construct M' from M one does the following: Let A_j be the column of M that is just to the left of the O that will be moved under stabilization. We copy A_j and call it A'_j and then add the new column to the right of A_j. Let B_j be the row right under the X and O to be changed in M.

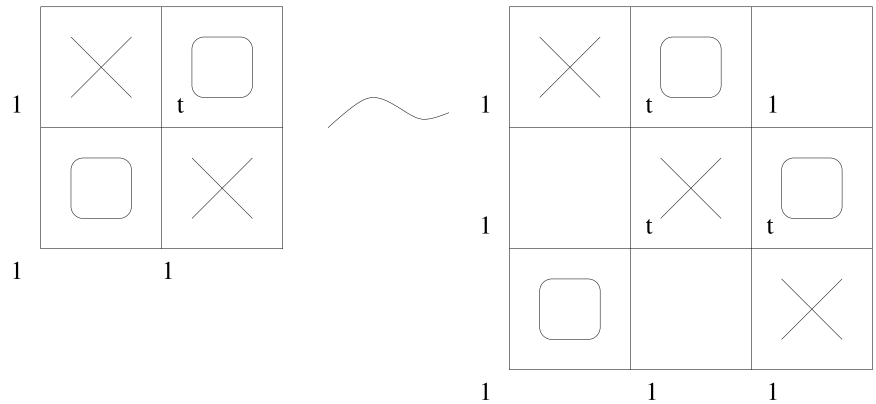


Figure 4: Stabilization of the Unknot

One makes a clone B_j' and places it right above B_j. Now we multiply the entry that is in the intersection of the new row and column by $t^{\pm 1}$ the sign depending on orientation. Now one calculates $\det M'$ by replacing A_j by A_j - A_j' which is just a column of zeroes with one entry $a_{ij}(t-1)$ where a_{ij} is an entry in M', specifically the intersection of the new row and column. If one calculates the determinant of M' by expanding along this column one obtains $\pm a_{ij}(1-t)\det M = \det M'$ from which the statement follows.

$\square$

An example of the above construction is given as follows: consider the winding matrix of a 2×2 grid diagram for the unknot

$$\begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}$$

Here

$$A_j = \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Copying A_j to the right gives the intermediate matrix

$$\begin{pmatrix} 1 & t & t \\ 1 & 1 & 1 \end{pmatrix}$$

For this matrix it is clear that

$$B_j = \begin{pmatrix} 1 & t & t \end{pmatrix}$$

Copying B_j to the top of the matrix gives

$$\begin{pmatrix} 1 & t & t \\ 1 & t & t \\ 1 & 1 & 1 \end{pmatrix}$$

Now the intersection of the new row and column is the t to the top right. This entry is the point that is to the immediate right of the segment joining the new X and O in the stabilized diagram. It follows that one multiplies this entry by t^{-1}. One finally obtains the winding matrix:

$$\begin{pmatrix} 1 & t & 1 \\ 1 & t & t \\ 1 & 1 & 1 \end{pmatrix}$$

Lemma 3. *Commutation changes the determinant by sign.*

Proof. It is clear that Commutation changes only one column of the winding matrix. Let C be that column in the original matrix and let R and L be the columns to the right and left respectively of C in the original matrix(See Figure 2). Let C' be the corresponding column in the new matrix. One sees that $R - C$ and $C' - L$ are the same column. □

One sees that these three lemmas show that the minesweeper determinant is an invariant of the knot if one takes into account the ambiguity of factors $(1 - t)$, $\pm t^{\pm n}$.

Define $n = \frac{1}{2}(I(X, X) - I(O, O) + N - 1)$, $\widetilde{\Delta}_K(T) = \frac{\Delta_K(T)}{(1-t)^{N-1}}$ where N is the size of the grid.

Lemma 4. $t^n\widetilde{\Delta}_K(T)$ *is an invariant up to sign.*

Proof. First consider the case of cyclic permutation. Consider a grid diagram D of a knot K and the corresponding diagram D' obtained by cyclic permutation. Define X_i as the number of Xs to the lower left of the X in the ith column. Then for D obtain $I(X, X) = \sum_{j=1}^{n-1} X_j + N - i$ where the X in the last column is on row i. For D' one obtains $I(X', X') = \sum_{j=1}^{n-1} X'_j + i - 1$. So $I(X, X) - I(X', X') = N - 2i + 1$. By a completely analogous argument $I(O, O) - I(O', O') = N - 2k + 1$ where the O in the last column is in row k.

So it follows that $n - n' = \frac{1}{2}(N - 2i + 1 - (N - 2k + 1)) = k - i$. Note that it was proved earlier that under cyclic permutation $\Delta_K(T)$ changes by multiplication by $\pm t^j$ for some integer j. If one looks at the argument more closely, one finds that this j must equal $k-i$ where k and i are the rows of the O and X in the last column. This is exactly what is needed to be shown. More formally, $t^{n'}\widetilde{\Delta}'_K(T) = t^{n-(k-i)}\widetilde{\Delta}'_K(T) = \pm t^n t^{-(k-i)}\widetilde{\Delta}_K(T)t^{k-i} = \pm t^n\widetilde{\Delta}_K(T)$.

For a grid diagram D of a knot K let D' be the grid diagram after commutation is used on D. Then $I(X, X) = \sum_i X_i$. Let X_k and X_{k+1} be the sums associated to the Xs that are moved in the operation. The only difference is that X_{k+1} picks up 1. So $I(X', X') = \sum_i X_i + 1$. It follows that $I(X, X) - I(X', X') = -1$ and similarly $I(O, O) - I(O', O') = -1$. Finally, $n - n' = \frac{1}{2}(-1 - (-1)) = 0$. Since commutation changes the minesweeper determinant only by sign this shows that this is invariant up to sign.

Let D' be the grid diagram after stabilization. Since invariance has been proved for the case of cyclic permutation without loss of generality assume that X_j is to the extreme upper left of the diagram(see Figure 5). As well, consider the number $r = a - b$ where $a =$ column number of O_j and $b =$ column number of X_j. It follows that $I(O', O') - I(O, O) = r$ and $I(X', X') - I(X, X) = r - 1$ One now sees that $n' - n = \frac{1}{2}[r - 1 - r + (N - (N - 1))] = 0$.

Now one verifies that $t^{n'}\frac{\Delta'_K(T)}{(1-t)^{N'-1}} = \pm t^n \frac{a_{ij}(1-t)\Delta_K(T)}{(1-t)^N}$ where a_{ij} is the entry as defined earlier in the paper. Note that it follows by how the diagram was permuted that $a_{ij} = 1$. With this information, one sees that $t^{n'}\frac{\Delta'_K(T)}{(1-t)^{N'-1}} = \pm t^n\frac{\Delta_K(T)}{(1-t)^{N-1}}$.

□

Now define $k = I(O, O) + 1$.

Lemma 5. *With n defined as before, $(-1)^k t^n\widetilde{\Delta}_K(T)$ is an invariant of a knot.*

Proof. It is not too hard to see that when one cyclically permutes a grid diagram D one changes the sign of the corresponding minesweeper determinant $N - 1$ times (one just exchanges the columns $N - 1$ times). What is key to the change of $I(O, O)$ after cyclic permutation is the O in the last column. To obtain the new sum one adds the number of Os above the last O and subtracts the number of Os below. If the number of Os below is i then the number above is $N - i - 1$.
More formally, $I(O, O) - I(O', O') = N - 2i - 1$.
So $(-1)^k t^n\widetilde{\Delta}_K(T) = (-1)^{N-2i-1+k'}t^n\widetilde{\Delta}'_K(T)(-1)^{N-1} = (-1)^{k'}t^{n'}\widetilde{\Delta}'_K(T)$.

For commutation, a comparison of D and D' yields $I(O, O) = I(O', O') - 1$. Since under commutation the determinant changes sign one obtains

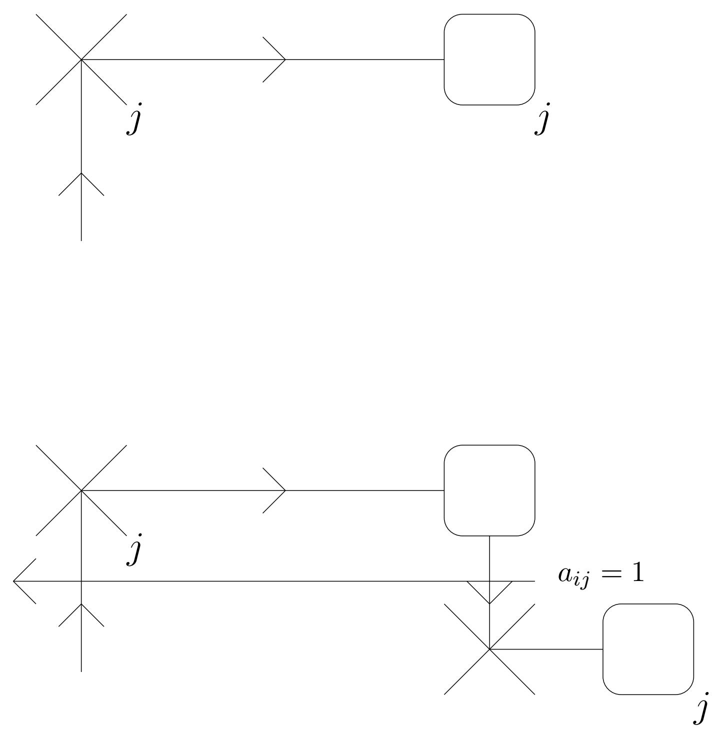


Figure 5: Here X_j is cyclically permuted to the top left of the diagram

$(-1)^k t^n \widetilde{\Delta}_K(T) = (-1)^{k'-1} t^{n'} \widetilde{\Delta}'_K(T)(-1) = (-1)^{k'} t^{n'} \widetilde{\Delta}'_K(T)$.

For stabilization, use cyclic permutation again to move O_j(see Figure 5) to the absolute bottom right of the diagram. Now note that $I(O', O') - I(O, O) = 0$. It was shown before that $\pm a_{ij}(t-1)\det D = \det D'$. The sign before depended on the factor $(-1)^{i+j}$. Now, note that $i = N-1$ and $j = N$ so in fact $-a_{ij}(t-1)\det D = \det D'$. The result follows.

$\square$

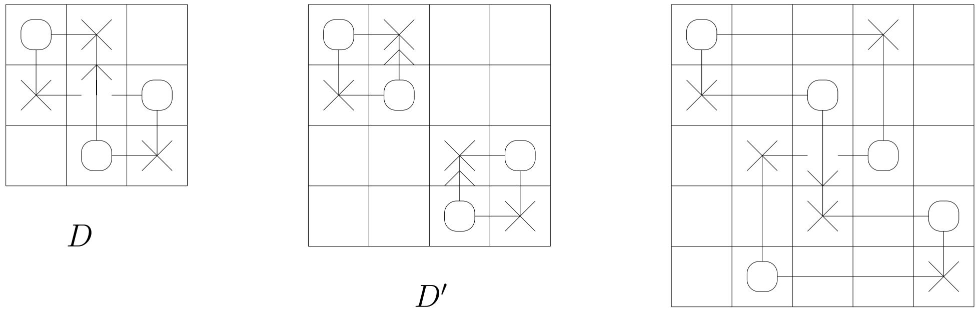
3 The Alexander Polynomial

Let F be a Seifert surface for a knot K. One can define a covering space X_∞ of $S^3 - K$ as follows:

Take a neighbourhood $F \times [0,1]$ of the surface and consider the space $Y = S^3 - (F \times [0,1])$. One sees that $\partial(S^3 - (F \times [0,1])) \cong (F \times 0) \cup (F \times 1) \cup \partial F \times [0,1]$. Now one takes countably many copies Y_i of Y and glues them together by identifying $F_i \times 1$ with $F_{i+1} \times 0$. Call the result X_∞. It can be shown that X_∞ does not depend on choice of F. The way to see this is to note that the fundamental group of the space X_∞ (that is the subgroup of the fundamental group that consists of loops in $S^3 - K$ that lift to loops in the cover) does not depend on F. This is so because a loop in the exterior X of a knot K must lift so that the basepoint is in the same copy of Y. This can only be true if and only if the loop intersects F zero times algebraically, which is equivalent to the linking number of the loop with K being zero. The latter statement shows that the group of X_∞ is dependent only on K which means by the theory of covering spaces that X_∞ does not depend on F. Now $\mathbb{Z}$ acts on the space X_∞ by translation, furthermore this induces an action of $\mathbb{Z}$ on $H_1(X_\infty; \mathbb{Z})$. So now it follows that $H_1(X_\infty; \mathbb{Z})$ is a $\mathbb{Z}[t, t^{-1}]$ module.

Now define the matrix A with a basis $[f_i]$ for $H_1(F; \mathbb{Z})$ $A_{ij} = lk(f_i^-, f_j) = lk(f_i, f_j^+)$. It is true that the module $H_1(X_\infty; \mathbb{Z})$ does have a square presentation matrix $tA - A^T$ where A is a Seifert matrix of the knot. The Alexander polynomial is found by taking the determinant of this matrix.

The Alexander polynomial is a well-known invariant of a knot. A goal of this paper is to show that the minesweeper determinant of a grid diagram along with the normalization factors detailed above give the so-called Conway-normalized Alexander polyomial. To do this, all one needs to do is to show that $(-1)^k t^n \Delta_K(T)$ satisfies the skein relation and $(-1)^k t^n \Delta_K(1) = 1$. The latter fact is trivial to verify and the former requires a bit of computation, which will be done in the next section.



D D' D''

Figure 6: The Grid Diagrams in the Example

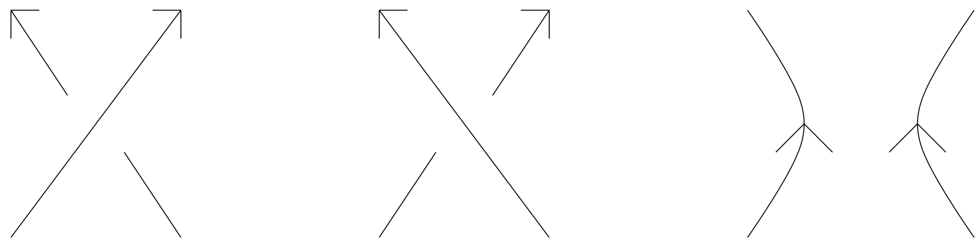
Since there are $2g$ generators for $H_1(F;Z)$ it follows that the Seifert matrix for a knot is a $2g \times 2g$ matrix. Thus the width of the polynomial is $\leq 2g$. It follows that the Alexander polynomial gives a lower bound for the genus.

4 The Skein Formula

The skein formula gives a way to compute the Alexander polynomial of a knot. If three knots K_+ K_- and K_0 are the same except in a neighbourhood of a point as detailed in the picture then they satisfy the relation $\Delta K_+(T) - \Delta K_-(T) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta K_0(T)$ [Lic97]. The skein relation for the minesweeper determinant takes a bit of computation to verify using the methods that have been used earlier in the paper. The author then will present an example, the way to generalize will be clear from the example. Let D be a 3×3 grid diagram for the unkot with winding matrix

$$\begin{pmatrix} 1 & t^{-1} & 1 \\ 1 & 1 & t \\ 1 & 1 & 1 \end{pmatrix}$$

Let D' be the grid diagram after changing D locally so that there are now no crossings in the diagram. One obtains the winding matrix for D'



K_+ K_- K_0

$$\begin{pmatrix} 1 & t^{-1} & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & t \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Let D'' be the grid diagram after changing D locally so that its crossing is reversed. One obtains the winding matrix for D''

$$\begin{pmatrix} 1 & t^{-1} & t^{-1} & t^{-1} & 1 \\ 1 & 1 & 1 & t^{-1} & 1 \\ 1 & 1 & t & 1 & 1 \\ 1 & 1 & t & t & t \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Obtain the matrix B_1 by multiplying rows $1-4$ of D'' by t. Thus $t^{-4}\det B_1 = \det D''$. Let C_i denote the ith column. Now replace C_3 by $C_3 - C_4$ and obtain

$$\begin{pmatrix} t & 1 & 0 & 1 & t \\ t & t & t-1 & 1 & t \\ t & t & t^2-t & t & t \\ t & t & 0 & t^2 & t^2 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

One now factors out $t-1$ from the C_3 to obtain $\det B_1 = (t-1)\det B_2$ where B_2 is the winding matrix

$$\begin{pmatrix} t & 1 & 0 & 1 & t \\ t & t & 1 & 1 & t \\ t & t & t & t & t \\ t & t & 0 & t^2 & t^2 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Let R_i denote the ith row. Now for B_2 if one replaces the R_3 with $R_3 - tR_2$ one obtains B_3 which is the matrix

$$\begin{pmatrix} t & 1 & 0 & 1 & t \\ t & t & 1 & 1 & t \\ t-t^2 & t-t^2 & 0 & 0 & t-t^2 \\ t & t & 0 & t^2 & t^2 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Now construct the matrix B_4 by dividing R_3 by $(t-1)$

$$B_4 = \begin{pmatrix} t & 1 & 0 & 1 & t \\ t & t & 1 & 1 & t \\ t & t & 0 & 0 & t \\ t & t & 0 & t^2 & t^2 \\ 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Now note that $\det D'' = -t^{-4}(1-t)^2 \det B_4$. As well, if one expands along C_3 one obtains the matrix B_5 which is

$$\begin{pmatrix} t & 1 & 1 & t \\ t & t & 0 & t \\ t & t & t^2 & t^2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

It is clear that $\det D'' = t^{-4}(1-t)^2 \det B_5$. Now for B_5 replace C_3 by $C_3 - C_4$ and obtain B_6

$$\begin{pmatrix} t & 1 & 1-t & t \\ t & t & -t & t \\ t & t & 0 & t^2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Recall D'. If one "normalizes"' D' by multiplying in the winding matrix $R_1 - R_3$ by t one obtains the matrix A_1

$$\begin{pmatrix} t & 1 & t & t \\ t & t & t & t \\ t & t & t & t^2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

It is clear that $\det D' = t^{-3} \det A_1$. Now if one takes C_3 and replaces it by $C_3 - C_2$ then factors out the $(t-1)$ in C_3 one obtains A_2

$$\begin{pmatrix} t & 1 & 1 & t \\ t & t & 0 & t \\ t & t & 0 & t^2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

It follows that $(t-1)\det A_2 = \det A_1$. Now notice that $\det B_6 - (1-t)\det A_2 = \det H_1$ where H_1 is the matrix

$$\begin{pmatrix} t & 1 & 0 & t \\ t & t & -t & t \\ t & t & 0 & t^2 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Setting H_2 to be the matrix

$$\begin{pmatrix} t & 1 & t \\ t & t & t^2 \\ 1 & 1 & 1 \end{pmatrix}$$

one sees that $t \det H_2 = \det H_1$. One notices as well that H_2 satisfies $t^{-2}\det H_2 = \det D$.

Putting this together one finds the expression $t^4 \det D'' - t^2 t(1-t)\det D'(1-t) = t^3 \det D(1-t)^2$. Since the minesweeper determinant is well-defined up to factors of $\pm t^n$, factors of $(1-t)$ an expression like this is what is expected. In fact, taking the normalized version of the Alexander polynomial yields the skein relation. To do this note that for D'' $n'' = 2$; for D' $n' = \frac{3}{2}$; for D $n = 1$ and k is even for all the grids. If one divides through the above expression by $(1-t)^4$ and notes that $t(1-t) = t^{\frac{3}{2}}(t^{\frac{-1}{2}} - t^{\frac{1}{2}})$ one obtains the formula

$t^2(\frac{t^2 \det D''}{(1-t)^4} - \frac{t \det D}{(1-t)^2}) = t^2(t^{\frac{-1}{2}} - t^{\frac{1}{2}})\frac{t^{\frac{3}{2}} \det D}{(1-t)^3}$.

Canceling the t^2 yields exactly the skein relation for the normalized Alexander polynomial. When one reverses or undoes a crossing there is only a small region that changes in the diagram. For the general proof one just focuses on this small region. Indeed, the general proof follows from mimicking the manipulations in the above detailed example plus keeping track of the change in the normalization factor t^n when going from D to D' to D''.

One may state as a corrollary that $(-1)^k t^n \widetilde{\Delta}_K(T)$ is the normalized Alexander polynomial.

5 The Genera of Torus Knots

A torus knot is a knot that can be laid out on the surface of an unknotted torus in $\mathbb{R}^3$ such that the knot does not intersect itself on the surface of the torus. Torus knots are generally identified with pairs of relatively prime integers (p, q), where p is the number of times the knot intersects a longitude curve on the torus and q is the number of intersections of the knot with a meridian curve on the torus. Seifert's algorithm gives a method for constructing a surface F from a knot K such that $\partial F = K$. Let K be a (p, q) torus knot. Then one can easily construct a closed braid representation of K with p strands by the word $(\sigma_1\sigma_2...\sigma_{p-1})^q$. When constructing F one sees that there must be p Seifert circles corresponding to the p strands, and that there must be $(p-1)q$ crossings. The Euler Characteristic is then $\chi = p - (p-1)q = -(pq - p - q)$. So in fact this gives $g = \frac{(p-1)(q-1)}{2}$. Thus Seifert's algortihm gives this upper bound for the genus of a torus knot.

Torus knots have a particularly nice grid diagram presentation. For example, to construct a grid diagram D of a (p, q) torus knot one may take a $p + q \times p + q$ toroidal grid diagram and put an X in every cell along the diagonal. Then there should be two parallel diagonals of Os on each side of the diagonal, one of length p and one length q respectively.

That this representation describes a p,q torus knot can be seen by reversing the connection of X's and O's so that there are no crossings on the grid diagram. The resulting diagram will intersect one edge of the diagram p times and the other q times. In joint work with Yael Degany and Andrew Freimuth over the summer, mostly grid diagrams of this form were investigated. The goal of this paper was to compute the genera of torus knots using their grid presentation and the Alexander gradings associated with the chain complex of the diagram. To do this we proved lemmas for grid diagrams that implied a nice simplification of the formula for the Alexander grading.

We say that a point on a grid diagram $\hat{a} = (k, a_k)$ has a weight $w(\hat{a}) = c - d$ where $c =$ the number of $Xs = (a, b)$ such that $a \geq k$, $b \geq a_k$, $d =$ the

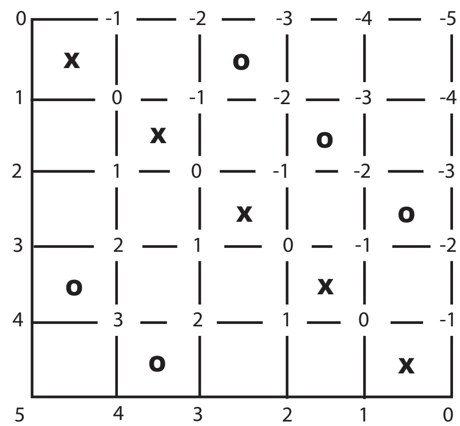


Figure 7: The weights for a trefoil

number of Xs such that $k > a$, $a_k > b$.

Lemma 6. *On any grid diagram if we have $w(\hat{a}) = r$, then $\hat{b} = (k+1, a_k)$ has weight $r-1$, similarly $\hat{c} = (k, a_k + 1)$ has weight $r-1$.*

Proof. Say $w(\hat{a}) = r$ where $\hat{a} = (k, a_k)$. So (k+1,a_k) is a shift over to the right of the diagram. In the column separating the two points there exists exactly one X=(a,b).There are two cases $b \geq a_k$ or $b < a_k$. In first case we get $w(k+1, a_k) = (c-1) - d = r-1$ and in the second $w(k+1, a_k) = c - (d+1) = r-1$. A similar argument can be made for $\hat{c}$, where the only difference in the weight must be one X in the row separating the points. $\square$

Figure 7 gives an example of a grid diagram with the weights filled in. The distribution of the weights solely depends on the size of the grid diagram; thus two $n \times n$ grid diagrams of two different knots will have the same weight distribution.

Lemma 7. *For any grid diagram $I(x, X) - I(X, x) = n$ where $x = Id$.*

Proof. $I(x, X) - I(X, x) = w(\hat{a}_1) + w(\hat{a}_2) + ... + w(\hat{a}_n) = n + (n-2) + (n-4) ... + (n - 2(n-1)) = n^2 - 2\sum_{i=1}^{n-1} i = n^2 - 2\frac{n(n-1)}{2} = n.$ $\square$

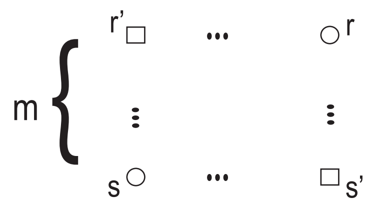


Figure 8: A detail for a grid diagram where the two generators x and y differ by a transposition

Lemma 8. *Let (ab) denote the transposition exchanging a and b. For a generator x if $I(x,X) - I(X,x) = n$ then $I((ab)x, X) - I(X,(ab)x) = n$.*

Proof. Let $y = (ab)x$. Then the intersection points that x and y do not have in common form a rectangle of length m on the grid. Let r be the weight of the upper right point of the rectangle and s be the weight of the lower left point where both belong to x. Then the upper left and lower right points r' and s' belong to y (See Figure 8). Then we have $s' = r+m$ and $r' = s-m$. So $I(y,X)-I(X,y) = I(x,X)-I(X,x)-r-s+r'+s' = I(x,X)-I(X,x)$. $\square$

The above lemmas imply that the Alexander grading simplifies to

$$A(x) = I(x,X) - I(x,O) + C$$

In fact, $C = -\frac{1}{2}(I(X,X)-I(O,O)+N-1)$, this is how the normalization n from section 2 can be derived.

The sum $I(x,X) - I(x,O) = I(X,x) - I(O,x)$ is just the sum of the winding numbers of all the points of the generator x and C is a quantity dependent only on the grid diagram and not the particular generator x. It is obvious that finding the maximum or minimim Alexander grading of all generators corresponds to finding the generators with the biggest or least sum of winding numbers.

So one is just concerned by the quantities

$$\begin{aligned} A_{res}(x) &= I(x,X) - I(x,O) \\ \text{or } A_{res}(x) &= I(X,x) - I(O,x) \end{aligned}$$

In the general grid diagram for a torus knot in 10, $A_{res}(x) = 0$ for all matchings in areas a and d. This is seen from the previous equations and

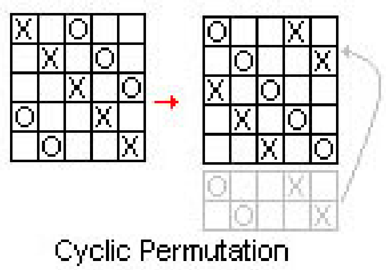


Figure 9: A Dynnikov type 3 shift

because any point in the region a will be above no X's or O's and in region d will be below no X's or O's. $A_{res}(x)$ will be negative for matchings in areas b and c because there are no X's and always O's below any x in area b and in area c any x is always below an O but never below any X. Therefore the maximum value of $A_{res}(x)$ is 0, and because of the structure of the grid diagram the unique matching where this is possible is forced to be the matching with points in the upper left corner of all O's. Since the grid diagram is a torus, it can be shifted upwards by a Dynnikov type 3 move, cyclic permutation, shown in 9. Then by similar reasons, the values of $A_{res}(x)$ around the center diagonal will be all positive, the values in the corners are all 0, forcing the unique matching with the least relative Alexander grading to be the matching with points in the upper left corner of all X's.

As all torus knots will have a grid diagram of the same general form, and the maximal and minimal Alexander grading can be computed in general terms for a $T_{p,q}$ torus knot in terms of p and q. Looking at each component of A(x) in terms of p and q gives the following formulas

$$\begin{aligned} A_{min}(x) &= \frac{1}{2}\Big[(p+q)-(p+q)(p+1) \\ &\qquad +2\sum_{i=1}^{min(p,q)} i+(p+q-1-2min(p,q))min(p,q) \\ &\qquad +pq+p+q-1\Big] \end{aligned}$$

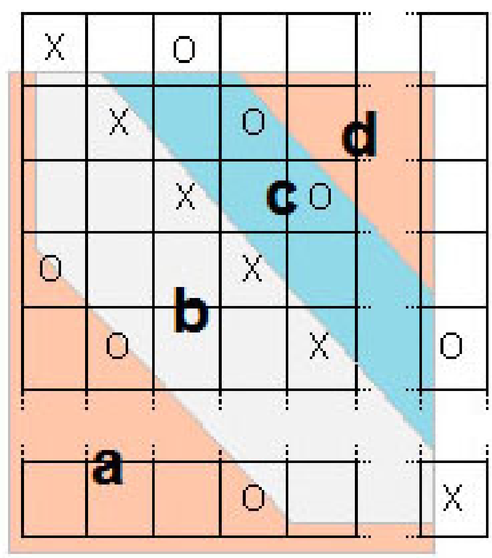


Figure 10: A general form of a grid diagram for a p,q torus knot

$$\begin{aligned}
\text{for } p > q,\ A_{max}(x) &= \frac{1}{2}\Big[p(q-1) + 2\sum_{i=1}^{q} i + p(p-q) - q(p+1) \\
&\qquad -p(q-1) + pq - (p+q-1)\Big] \\
\text{for } p < q,\ A_{max}(x) &= \frac{1}{2}\Big[p(q-1) + 2\sum_{i=1}^{p} i + (q-p)(p+1) - q(p+1) \\
&\qquad -p(q-1) + pq - (p+q-1)\Big]
\end{aligned}$$

These simplify to

$$\begin{aligned}
A_{min}(x) &= \frac{-pq - p - q + 1}{2} \\
A_{max}(x) &= \frac{pq - p - q + 1}{2} = \frac{(p-1)(q-1)}{2}
\end{aligned}$$

It has already been discussed that the width of the Alexander polynomial gives a lower bound for the genus. The above computation gives the width of the Alexander polynomial. This shows that the genus of a (p, q) torus knot is $\frac{(p-1)(q-1)}{2}$.

If one tries to use these methods for most other knots one runs into trouble because their grid diagram presentations are not so "nice". In these cases it is not too to see what generator has the maximal grading. However, it might be possible that there are other types that are "nice" in the sense they are susceptible to the above methods. Possible candidates include positive braids.

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