THE BIG PROBLEM
FALL 2008

MATH W4051 WITH R. LIPSHITZ

1. About the big problem

This assignment is a bit like a midterm paper. In undergraduate mathematics courses, one rarely gets to struggle with a problem for more than a week. By contrast, when doing mathematics research one generally works on a given problem for several years. This assignment will give you a small sense of what that feels like. You will choose a problem (from a list of suggestions, below) that is harder than what you see on problem sets. You'll then go through various stages of thinking about the problem at the end of which you'll turn in a complete, carefully written solution. (Your solution will probably be about four pages.)

Don't panic. I think most of you will be able to solve one of these problems, perhaps with a nudge or two in the right direction. If, after struggling, you can't solve the one you chose, that's fine. In that case, you'll write (a) an account of your struggles (what you tried, where you got stuck) and (b) an exposition of a proof which you read somewhere (citing the sources you use, of course). Learning to find, read, and write about mathematics is a valuable skill, too.

Finally, this is an experiment. So it may be a disaster. But I'm optimistic it won't be—and that you'll enjoy it.

2. Time line

- September 2–9: Choose a problem. On September 9, turn in to me your problem choice.
- September 9–16: Think about your problem in your spare moments. Make sure you understand its statement. Try to understand why it's (a) true and (b) not completely obvious. If you're having trouble with this, switch problems (at most once). Talk about your problem with your friends. Explain it to someone as far from mathematics as you can.
- September 16–23: Write a strategy for solving your problem. This should look something like: "If I knew A, I would have the solution. It seems that B might imply A. I think B is true, or at least I can't think of a counterexample. Under the following hypothesis, I can prove B. Maybe I can remove these hypothesis by doing C." (If you're stuck at this stage, talk to me or Tom for help.) Look in the literature for helpful lemmas (though preferably not for a solution to your particular problem).
- September 23–30: Refine your strategy. Replace parts that turn out to be false. Write proofs for the parts of your strategy which you can. (If you're stuck at this stage, talk to me or Tom for help.)
- September 30–October 7: Finish solving your problem, by filling-in the rest of the steps in your strategy. (If you're still stuck, do a literature search (in the library) for the solution. You'll write an exposition, in your own words, of the proof you find—with complete citations, of course.)
- October 7–21: Write up your solution. You will probably find some serious holes in your argument. Keep track of them. Work to try to fix them. Get help when you need it.
- October 21–28: Revise your draft. Read it once each for:
  - Correctness.
  - Style and readability.
  - Grammar.
- October 28: Turn in the first draft of your solution.
- November 6: You'll get back your first draft, with comments. Revise thoroughly.
- November 13: Turn in the final draft of your solution.

3. Problems to choose from

These problems are not all of similar difficulty. My feeling is that in order of increasing difficulty they are Problem 6, 3, 4, 1, 5, and 2. Keep this in mind when choosing one, but also try to choose one that catches your interest.

Problems 1 and 2 use a little real analysis, about approximating continuous functions by smooth or piecewise-linear functions. This is not very hard (and is discussed, say, in Rudin's *Principles of Mathematical Analysis*), but if you haven't taken any real analysis, you might want to avoid them.

Problem 1. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be a continuous map, and $p \in \mathbb{R}^2 \setminus \gamma(S^1)$ a point not in the image of γ. Then γ has a well-defined *winding number around* p, denoted $W(\gamma, p) \in \mathbb{Z}$. Intuitively, this is the number of times γ winds counterclockwise around p. Note that this number should be unchanged under deformations of γ in the complement of p.

One way of defining $W(\gamma, p)$ is in terms of the circulation of a certain vector field around p. Make this precise. A suggested rough outline:

- Find a vector field on $\mathbb{R}^2 \setminus \{p\}$ as described. (Hint: you want it to be conservative... why?)
- Define the winding number for any smooth (infinitely differentiable) curve in $\mathbb{R}^2 \setminus \{p\}$. (Why is this easier than doing it for any continuous path?)
- Define the winding number in general, by approximating any continuous path with a smooth path.

(A variant on this problem, if you've taken complex variables, is to use some ideas from that for the first two steps.)

Problem 2. Intuitively, the winding number can also be defined as follows: choose a ray R from p out to infinity (in any direction). Then, roughly speaking, the winding number of γ around p is the number of times R intersects γ (counted with sign).

- Think about what the problem means. What do I mean by "counted with sign"? Why did I say "roughly speaking"? (Why is the number I defined not well-defined?)
- Give a correct definition of the winding number for piecewise linear curves (curves made of a finite number of line segments). Prove your definition is well-defined. (You might want to add some additional hypotheses on your paths.)
- Approximate an arbitrary curve by a piecewise-linear one, and use this to define the winding number in general.

Problem 3. The Brouwer Fixed Point Theorem states: Let $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then there is some point $p \in \mathbb{D}^2$ such that $f(p) = p$. (Such a point p is called a *fixed point* of f.)

The Brouwer Fixed Point Theorem can be deduced from *Sperner's Lemma*, which states: let T be a triangle, with vertices labeled R, G and B. Divide T up into little triangles (each with three vertices on its boundary). Color each

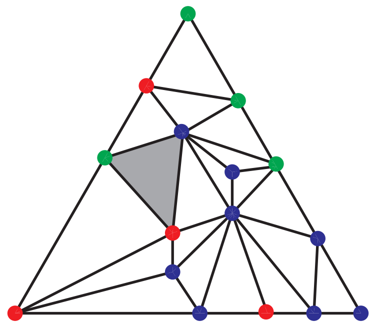

Figure 1. An illustration of Sperner's lemma.

vertex in this triangulation either red, green or blue, with the requirement that:

- R is red, G is green and B is blue.
- Every vertex along the edge of T between R to G is colored red or green; every vertex along the edge of T between G to B is colored green or blue; and every vertex along the edge of T between B to R is colored blue or red.

Then there is a small (undivided) triangle with one vertex of each color.

(See Figure 1 for an illustration.)

In this problem, you should:

- Prove Sperner's Lemma.
- Use Sperner's Lemma to prove the Brouwer Fixed Point Theorem. (Hint: $\mathbb{D}^2$ is homeomorphic to a triangle. If $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ and $p \in \mathbb{D}^2$, and v is a vertex, either $f(p)$ is closer to v than p or it isn't.)
- Optional: can you generalize this to dimensions bigger than 1?
- Also optional: use Sperner's lemma to prove the *fundamental theorem of algebra*.

Remark. It's easy to find the proofs from this problem on the web or in a book. Please don't do that before thinking hard about it for several weeks.

Problem 4. Let P be a polyhedron. That is, P is a space constructed by gluing together a finite number of polygons along edges such that the result is homeomorphic to the sphere S^2. Let v be the number of vertices of P, e the number of edges, and f the number of faces. Then $v - e + f = 2$. (This is called Euler's formula.) Prove this.

Suggestions:

- Formulate the statement in terms of graphs in the plane.
- Prove that if G and G' are graphs in the plane, such that G' is obtained by subdividing G, then the statement is true for G' if and only if it is true for G.
- Prove that if G and G' are any two planar graphs then there is a graph G'' which is a subdivision of both G and G'. Conclude the result.

If you like, formulate an analogous theorem for other surfaces, and prove that if you can.

Problem 5. The Jordan Curve Theorem states: Let $\gamma\colon\ S^1 \to \mathbb{R}^2$ be an injective, continuous map. Then $\mathbb{R}^2 \setminus \gamma(S^1)$ has exactly two connected components, one bounded and the other unbounded.

(A map γ as in the statement of the theorem is often called a *simple closed curve* or sometimes a *Jordan curve*.)

Prove the Jordan curve theorem under the assumption that γ is piecewise-linear (made up of a finite number of line segments). (You might like to adapt ideas from Problem 2 for this. Or you might not.)

Alternatively, prove the Jordan curve theorem under the assumption that γ is smooth, using ideas from Problem 1.

Problem 6. This problem is about using several different ideas to prove various different spaces are not homeomorphic.

- Prove that $\mathbb{R}^2$ is not homeomorphic to S^2. (Find a topological property that distinguishes them.)
- Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. (What happens if you delete a point?)
- Which letters of the alphabet are homeomorphic? (It depends on the font a bit, maybe; choose one.)
- Intuitively, $\mathbb{R}$ has two "ends". $\mathbb{R}^2$ has a single end. $\mathbb{R}^2 \setminus \{(0,0)\}$ has two ends. (For argument, let's say the half-open interval $[0,1)$ has just one end, 1.) Make the notion of having one end precise, and use it to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^2 \setminus \{(0,0)\}$. Make the notion of an end of a topological space precise. (This is, by far, the hardest part of the problem.)
- With your definition, can you find a subspace of the plane with infinitely many ends? Uncountably many ends?
- On a different note, how many homeomorphism types of open subsets of $\mathbb{R}$ are there? How about open subsets of $\mathbb{R}^2$?
- Think of an interesting question along the lines of something in this problem, and solve it, too.