MATH W4051 PROBLEM SET 12
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(1) Recall from problem set 3:q The group $\mathbb{Z}/5$ acts on S^3 as follows. Write $S^3 = \{(z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $\mathbb{Z}/5 = \{1, \zeta, \zeta^2, \zeta^3, \zeta^4\}$. Then define

$$\zeta^n(z,w) = (e^{2\pi i n/5} z, e^{6\pi i n/5} w).$$

The quotient S^3/G is called the *lens space* $L(5,3)$.

(a) Prove that S^3 is the universal cover of $L(5,3)$. (The work here is proving that the map $S^3 \to L(5,3)$ is a covering map.)

(b) Use covering space theory to compute $\pi_1(L(5,3))$.

(c) Optional: generalize this to define lens spaces $L(p,q)$ for any relatively prime p and q. Use covering spaces to compute $\pi_1(L(p,q))$.

(2) Let $p\colon \tilde{X} \to X$ be a covering space. Prove: if $\tilde{X}$ is compact then for any $x \in X$, $p^{-1}(x)$ is finite.

(3) Find a 2-fold cover of the Möbius strip by an orientable surface. Find a 2-fold cover of the Klein bottle by an orientable surface.

(4) In this problem, you'll use the theory of covering spaces to prove a theorem in algebra.

(a) Let X be a graph, i.e., a 1-dimensional CW complex. Let $\tilde{X}$ be a covering space of X. Prove that $\tilde{X}$ is also a graph, i.e., a (not necessarily finite) 1-dimensional CW complex.

(b) Prove that the fundamental group of any graph is a free group.

(c) Use the correspondence between covering spaces and subgroups of π_1 to conclude that if G is a free group and H is a subgroup of G then H is also a free group. *Remark.* This can be proved by purely algebraic means, but it's not easy.

(5) In spite of the previous problem, free groups can have some unexpected subgroups...

(a) Draw a 2-fold cover of the figure 8, and the map from it to the figure 8. (Hint: it will have three holes.) Use it to observe that F_3 is a subgroup of F_2. Describe this subgroup explicitly, in terms of generators.

(b) Similarly, find a subgroup of F_2 isomorphic to F_4.

(c) Similarly, find a subgroup of F_2 which is free on infinitely many generators. (*Hint for all parts.* If you get stuck, you'll probably find the pictures on p. 58 of Hatcher helpful.)

Optional. (If you've taken a course in differential geometry...) Prove that if M is a non-orientable smooth manifold then M has a 2-fold cover by an orientable manifold. (If you want a hint, ask.)

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