MATH W4051 PROBLEM SET 11
DUE NOVEMBER 26, 2008.

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(1) We used the following group theory fact in class, to show there was a surjective homomorphism from the fundamental group of the trefoil complement to D_3:

Let $G = \langle g_1, g_2, \dots, g_k \mid r_1, r_2, \dots, r_l \rangle$ be a group given in terms of generators and relations. Write $r_i = g_{i,1}^{n_{i,1}} g_{i,2}^{n_{i,2}} \dots g_{i,j_i}^{n_{i,j_i}}$.

Let H be any group, and $h_1, \dots, h_k \in H$. Then there is a group homomorphism $f\colon G \to H$ such that $f(g_i) = h_i$ ($i = 1, \dots, k$) if and only if, for all

$$h_{i,1}^{n_{i,1}} h_{i,2}^{n_{i,2}} \dots h_{i,j_i}^{n_{i,j_i}} = 1_H$$

for $i = 1, \dots, l$.

Prove this. (Hint: one direction is easy. For the other, you'll use the definition of G as a quotient group of a free group.)

(2) Let $\mathbb{R}P^2$ denote the space of lines through the origin in $\mathbb{R}^3$. That is, $\mathbb{R}P^2 = (\mathbb{R}^3 \setminus 0)/(p \sim \lambda p)$ (where $\lambda \neq 0$). Compute $\pi_1(\mathbb{R}P^2)$.

(Hint: you can either do this directly using van Kampen's theorem or by putting a cell structure on $\mathbb{R}P^2$. To do the former, let U be a neighborhood of the vertical line $(\{(0,0,\lambda)\})$, and V the complement of a smaller neighborhood of this line.)

(3) Let $\mathrm{Homeo}(X)$ denote the group of homeomorphisms $X \to X$ and $\mathrm{Homeo}(X, x_0)$ denote the subgroup of homeomorphisms $X \to X$ sending x_0 to x_0.

  (a) Prove that $\mathrm{Homeo}(X, x_0)$ acts on $\pi_1(X, x_0)$. Prove that if homeomorphisms f and g are homotopic rel x_0 then f and g act in the same way. (This says that the action descends to an action of the group of path components of $\mathrm{Homeo}(X, x_0)$, where we endow $\mathrm{Homeo}(X, x_0)$ with the compact-open topology.)

  (b) Let $H_1(X) = \pi_1(X, x_0)/[\pi_1, \pi_1]$ denote the abelianization of $\pi_1(X, x_0)$. Prove that $\mathrm{Homeo}(X)$ acts on $H_1(X)$, and that homotopic homeomorphisms act in the same way.

  (c) Find a homeomorphism $T^2 \to T^2$ which is not homotopic to the identity map, and prove that it is not homotopic to the identity map.

(4) Optional: recall the definition of $L(5, 3)$ from problem set 3. Compute $\pi_1(L(5, 3))$.

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