MATH W4051 PROBLEM SET 10
DUE NOVEMBER 19, 2008.

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(1) Computing π_1 through clever homotopy equivalences...
   (a) Let $S \subset \mathbb{R}^3$ consist of n lines through the origin. What is $\pi_1(\mathbb{R}^3 \setminus S)$? Prove (explain) your answer.
   (b) Let $S \subset \mathbb{R}^3$ consist of n parallel lines. What is $\pi_1(\mathbb{R}^3 \setminus S)$? Prove (explain) your answer.
   (c) Let $S \subset \mathbb{R}^3$ consist of n points. What is $\pi_1(\mathbb{R}^2 \setminus S)$? Prove (explain) your answer.
   (d) Let S denote the unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$. What is $\pi_1(R^3 \setminus S)$? Prove (explain) your answer.

(2) Prove the following strengthening of the Brouwer fixed point theorem: let $f\colon \mathbb{D}^2 \to \mathbb{R}^2$ be a map such that $f(S^1) \subset \mathbb{D}^2$. Then f has a fixed point $p \in \mathbb{D}^2$.

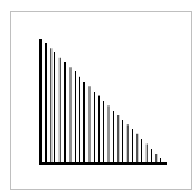
(3) Consider the letters of the alphabet, in the font shown:

a b c d e f g h i j k l m n o p q r s t u v w x y z

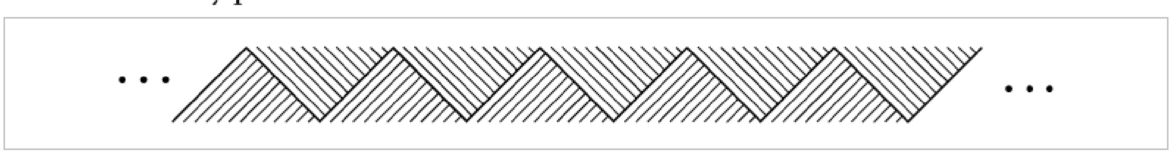
   (a) Group the letters into homotopy equivalence classes. Explain (briefly) your answers.
   (b) Group the letters into homeomorphism classes. Explain (briefly) your answers.

(4) Hatcher, exercise 6(a,b), p. 18, quoted here:

6. (a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1]\times\{0\}$ together with all the vertical segments $\{r\}\times[0,1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1]\times\{0\}$, but not to any other point. [See the preceding problem.]



(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.



(5) What is the product in the category of abelian groups? The coproduct? (Prove your answers, one of which might surprise you.)

(6) Here's another abstract description of free groups. The free group F_n on n symbols $a_1, \dots, a_n$ is characterized as follows: there is a map *of sets* $i\colon \{a_1, \dots, a_n\} \to F_n$, and for any group G and map *of sets* $f\colon \{a_1, \dots, a_n\} \to G$ there is a unique map $g\colon F_n \to G$ such that $f = g \circ i$.

In terms of diagrams:

$$\begin{tikzcd}
\{a_1,\dots,a_n\} \arrow[r, "i"] \arrow[dr, "f"'] & F_n \\
& G \arrow[u, dashed, "g"]
\end{tikzcd}$$

(a) Prove that this property characterizes F_n up to unique isomorphism. That is, given any two groups E and F and maps $i_E\colon \{a_1,\dots,a_n\} \to E$ and $i_F\colon \{a_1,\dots,a_n\} \to F$ satisfying the condition given above there is a unique isomorphism $f\colon E \to F$ so that the following diagram commutes:

$$\begin{tikzcd}
& E \\
\{a_1,\dots,a_n\} \arrow[ur, "i_E"] \arrow[dr, "i_F"'] & \\
& F \arrow[uu, dashed, "f"]
\end{tikzcd}$$

(b) Explain briefly that $\mathbb{Z}$ has this property for $n = 1$, so $\mathbb{Z} \cong F_1$.
(c) Explain why F_2, as defined in class, has this property for $n = 2$. (You may use either the construction in terms of words or the definition as a coproduct.)

Optional but particularly encouraged:

(6) Pushouts and amalgamated products. Given a category $\mathcal{C}$, objects $X, Y, Z \in ob(\mathcal{C})$ and morphisms $i_X\colon Z \to X$, $i_Y\colon Z \to Y$, a *pushout* of (X, Y, Z, i_X, i_Y) is an object P together with maps $j_X\colon X \to P$, $j_Y\colon Y \to P$ so that $j_X \circ i_X = j_Y \circ i_Y$ and, further, so that for any other object Q and maps $f\colon X \to Q$, $g\colon Y \to Q$ satisfying $f \circ i_X = g \circ i_Y$ there is a unique map $h\colon P \to Q$ so that $f = h \circ j_X$ and $g = h \circ j_Y$. In terms of diagrams: we have

$$\begin{tikzcd}
Z \arrow[r, "i_Y"] \arrow[d, "i_X"] & Y \arrow[d, "j_Y"] \\
X \arrow[r, "j_X"] & P
\end{tikzcd}$$

and, further,

$$\begin{tikzcd}
Z \arrow[r, "i_Y"] \arrow[d, "i_X"] & Y \arrow[d, "j_Y"] \arrow[ddr, bend left, "g"] & \\
X \arrow[r, "j_X"] \arrow[drr, bend right, "f"'] & P \arrow[dr, dashed, "h", "\exists!"'] & \\
& & Q
\end{tikzcd}$$

(This looks very complicated. You'll see from some examples that it isn't so complicated.)
(a) Prove that if a pushout exists then it is unique up to unique isomorphism.
(b) What is the pushout in the category of sets? (Hint: think first about the case when $Z = X \cap Y$.)
(c) What is the pushout in the category of topological spaces? (Hint: similar to sets.)

(d) The pushout in the category of groups is called the *amalgamated product.* Give an explicit description of it, and sketch a proof that it is, indeed, the pushout. (Hint: it's a quotient of the free product (a.k.a. coproduct).)

More **optional** problems:

(7) Give a definition of the product of arbitrarily many objects in a category. Check that the product topology is the product in the category of topological spaces.

(8) What's the product in the category of based topological spaces?

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