# MATH V2020 PROBLEM SET 6
# DUE OCTOBER 21, 2008.

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Problem 6 corrected in this version. Malapropism in statement of 2(a) corrected (eigenvalue vs. eigenvector).

(1) Use Cramer's rule to compute the inverse of:
   (a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
   (b) $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$.

(2) Eigenspaces...
   (a) Let $F\colon V \to V$ be linear. Prove that if $v$ is an eigenvector of $F$ of eigenvalue $\lambda$ and $a \in \mathbb{F}$, $a \neq 0$ then $(av)$ is also an eigenvector of $F$ of eigenvalue $\lambda$. (The proof should be short and needs few words.)
   (b) Let $F\colon V \to V$ be linear, and $v, w$ eigenvectors of $F$ of eigenvalue $\lambda$. Prove that $v + w$ is an eigenvector of $F$ of eigenvalue $\lambda$. (Similarly short and terse.)
   (c) It is *not* true that if $v$ and $w$ are eigenvectors with different eigenvalues then $v + w$ is an eigenvector. Give an example illustrating this. (Optional: prove that if $v$ and $w$ are eigenvectors with different eigenvalues then $v + w$ is *not* an eigenvector.)

(3) Find the eigenvalues of the following matrices. Find an eigenvector corresponding to each eigenvalue.
   (a) $\begin{pmatrix} 5 & -2 \\ 3 & 0 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

(4) The numbers in problems (3) and (5) were chosen so that the eigenvalues you found were integers. How did (or could) I do this? (Note: this is more of a pain than finding matrices which row-reduce nicely.)

(5) Let $A = \begin{pmatrix} -12 & 22 \\ -11 & 21 \end{pmatrix}$. Compute $A^{100}$ *by hand,* to ten significant digits.

(6) Recall that the *Fibonacci numbers* are defined by: $F_0 = 0$, $F_1 = 1$ and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. So, the first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55$.
   (a) Let
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
   Prove that the entries of $A^n$ are
$$\begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}.$$

(Hint: you want to use induction. Prove the result is true for $n = 1$ (trivial). Then, prove that if it holds for $n$ then it holds for $n + 1$.)

(b) Find the eigenvalues of $A$. Find an eigenvector corresponding to each eigenvalue. (This computation is a bit annoying.)

(c) Find a matrix $P$ so that $P^{-1}AP$ is diagonal.

(d) Use part 6c to compute $A^n$. This gives you another formula for $F_n$. What is it? (Again, a slightly annoying computation.)

(e) What is $\lim_{n\to\infty} F_{n+1}/F_n$?

(7) Prove that there is no invertible matrix $P$ so that $P^{-1}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} P$ is diagonal.

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