MATH V2020 PROBLEM SET 5
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INSTRUCTOR: ROBERT LIPSHITZ

Updated: I've made the first problem easier by adding a hypothesis, dropped Cramer's rule from this problem set (deleting last problem and modifying second to last), added an instruction to compute one of the determinants in Problem 7 by row-reduction.

(1) Let $F, G\colon V \to V$ be linear maps, and assume that G is invertible. Using the definition in terms of volume functions, prove that $\det(F \circ G) = \det(F)\det(G)$. (Hint: your proof should start "Let Vol be a volume form on V and $\mathcal{B} = [v_1, \dots, v_n]$ a basis for V." The proof should not be more than three or four mathematical "sentences.")

(2) Let $F\colon V \to V$ be a linear transformation and $\lambda \in \mathbb{F}$. Let $n = \dim(V)$. Prove that $\det(\lambda F) = \lambda^n \det(F)$. (Here, λF denotes the linear transformation $(\lambda F)(v) = \lambda(F(v))$.) (Hint: start the same way as the previous one. Your proof should again be short.)

(3) Suppose that A and B are $n \times n$ matrices, and $AB = I$. We proved in class that this implies A is invertible. Prove this again, using the determinant. (Hint: the proof should again be very short.)

(4) Prove that if A and B are similar $n \times n$ matrices then $\det(A) = \det(B)$. (Hint: short again. Remind yourself the definition of "similar matrices.")

(5) Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 2 & \pi & e \\ 0 & 0 & 3 & 7 & 17 \\ 0 & 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

by expansion by minors. Explain why your answer makes sense from the point of view of volume functions. (You might like to use an analogous 2×2 or 3×3 matrix to make your point.)

(6) An $n \times n$ matrix A is invertible if and only if A^T is invertible.
   (a) Prove this using determinants (one sentence).
   (b) Prove this directly, using the fact that $(AB)^T = B^T A^T$. (Roughly three sentences.)

(7) Compute the following determinants:
   (a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & -2 & -1 \end{pmatrix}.$$

Do this one both by expansion by minors and by row reduction.

(b)

$$\begin{pmatrix} 0 & 5 & 0 & 0 & 0 \\ 12 & 11 & 2 & 0 & 10 \\ 3 & 13 & 0 & 0 & 14 \\ 9 & 8 & 6 & 1 & 7 \\ 0 & 15 & 0 & 0 & 4 \end{pmatrix}$$

(c)

$$\begin{pmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & 0 & 0 \\ 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & 0 & 0 & i & j \\ 0 & 0 & 0 & 0 & k & l \end{pmatrix}.$$

(8) Is it usually faster to compute a determinant of an $n \times n$ (for n large) matrix by expanding by minors immediately or by row reducing first? Roughly how many arithmetic operations does each take? (You don't have to "prove" your answers; just explain them. But try to be precise and complete; this might take a couple of drafts.)

E-mail address: `rl2327@columbia.edu`