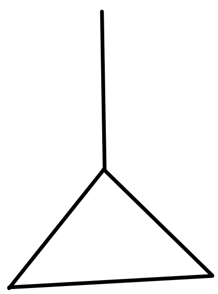
MATH W4052 PROBLEM SET 4
DUE FEBRUARY 14, 2011.

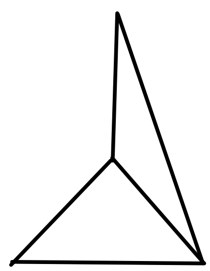
INSTRUCTOR: ROBERT LIPSHITZ

(1) Show that the standard k-simplex Δ^k is homeomorphic to
$$\{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid \forall i x_i \geq 0 \text{ and } x_1 + \cdots + x_k \leq 1\}.$$
(Hint: this is easy.)

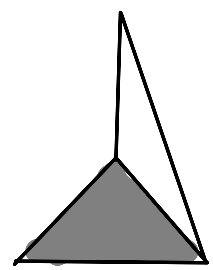
(2) Compute the simplicial homology of the following simplicial complexes:



(a)
(4 vertices, 4 edges)



(b)
(4 vertices, 5 edges)



(c)
(4 vertices, 5 edges, 1 face)

(3) (a) Find a simplicial complex $X_\bullet$ whose geometric realization $|X_\bullet|$ is the Möbius band. Write $X_\bullet$ as an abstract simplicial complex (i.e., as a set of vertices, a set of pairs of vertices, and a set of triples of vertices) and draw $|X_\bullet|$.
(b) Write down the simplicial chain complex $C_*(X_\bullet)$ (i.e., the three nontrivial groups $C_0(X_\bullet)$, $C_1(X_\bullet)$ and $C_2(X_\bullet)$ and the maps ∂_1 and ∂_2 between them).
(c) Compute the simplicial homology groups $H_*(X_\bullet)$ (i.e., $H_0(X_\bullet)$, $H_1(X_\bullet)$ and $H_2(X_\bullet)$).

(4) Compute the homology $(\ker(\partial_i)/\mathrm{Im}(\partial_{i+1})$ for each i) for the following chain complexes:
(a) $C_1 = \mathbb{Z}^3$, $C_0 = \mathbb{Z}^3$, $\partial_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$, $C_i = \{0\}$ for $i \notin \{0, 1\}$.
(b) $C_2 = \mathbb{Z}\langle f \rangle$, $C_1 = \mathbb{Z}\langle e_1, e_2 \rangle$, $C_0 = \mathbb{Z}\langle v \rangle$, $C_i = \{0\}$ for $i > 2$, $\partial_1(e_i) = 0$ $(\forall i)$, $\partial_2(f) = e_1 + e_2$.
(c) $C_2 = \mathbb{Z}\langle f_1, f_2 \rangle$, $C_1 = \mathbb{Z}\langle e_1, e_2 \rangle$, $C_0 = \mathbb{Z}\langle v \rangle$, $C_i = \{0\}$ for $i > 2$, $\partial_1(e_i) = 0$ $(\forall i)$, $\partial_2(f_1) = e_1 + e_2$, $\partial_2(f_2) = e_1 - e_2$.

(5) In this problem we'll define the *relative homology* of a simplicial complex relative to a subcomplex. We will use this later to put Seifert surfaces in a more general context.

(a) Given a simplicial complex $X_\bullet = \{X_0, X_1, \dots\}$, a *subcomplex* $Y_\bullet$ *of* $X_\bullet$ is a collection of subsets $Y_0 \subset X_0, Y_1 \subset X_1, \dots$ so that $\{Y_0, Y_1, \dots\}$ is itself a simplicial complex.
Let $X_\bullet$ be the simplicial complex $X_0 = \{v_0, v_1, v_2\}$, $X_1 = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$, $X_2 = \{\{v_0, v_1, v_2\}\}$ (so $|X_\bullet|$ is the 2-simplex). List all subcomplexes of $X_\bullet$.

(b) Suppose $X_\bullet$ is a simplicial complex and $Y_\bullet$ is a subcomplex of $X_\bullet$. Then for each i, $C_i(Y_\bullet) \subset C_i(X_\bullet)$. Observe that $\partial_i : C_i(X_\bullet) \to C_{i-1}(X_\bullet)$ has the property that $\partial_i(C_i(Y_\bullet)) \subset C_{i-1}(Y_\bullet)$. Using this, prove that ∂_i induces a well-defined map $\partial_i^{X/Y} : C_i(X_\bullet)/C_i(Y_\bullet) \to C_{i-1}(X_\bullet)/C_{i-1}(Y_\bullet)$ so that the following diagram commutes:

$$\begin{tikzcd} C_i(X_\bullet) \arrow[r, "\partial_i"] \arrow[d] & C_{i-1}(X_\bullet) \arrow[d] \\ C_i(X_\bullet)/C_i(Y_\bullet) \arrow[r, "\partial_i^{X/Y}"] & C_{i-1}(X_\bullet)/C_{i-1}(Y_\bullet). \end{tikzcd}$$

Here, the vertical arrows are the canonical quotient maps.
Also, prove that $\partial_i^{X/Y} \circ \partial_{i+1}^{X/Y} = 0$.
(Hint: there's not much work to do.)

(c) Given a simplicial complex $X_\bullet$ and a subcomplex $Y_\bullet$, the *relative homology groups* of the pair $(X_\bullet, Y_\bullet)$ are

$$H_i(X_\bullet, Y_\bullet) = \ker(\partial_i^{X/Y})/\mathrm{Im}(\partial_{i+1}^{X/Y}).$$

Compute the relative homology of the following pairs of simplicial complexes:

(i) $X_0 = \{v_0, v_1\}$, $X_1 = \{\{v_0, v_1\}\}$, $X_i = \emptyset$ for $i > 1$ (so $|X_\bullet|$ is an interval); $Y_0 = \{v_0\}$, $Y_i = \emptyset$ for $i > 0$.

(ii) $X_0 = \{v_0, v_1\}$, $X_1 = \{\{v_0, v_1\}\}$, $X_i = \emptyset$ for $i > 1$ (so $|X_\bullet|$ is an interval); $Y_0 = \{v_0, v_1\}$, $Y_i = \emptyset$ for $i > 0$.

(iii) $X_0 = \{v_0, v_1, v_2\}$, $X_1 = \{\{v_0, v_1\}, \{v_0, v_2\}\}$, $X_i = \emptyset$ for $i > 1$ (so $|X_\bullet|$ is an interval, with a vertex in the middle); $Y_0 = \{v_0, v_1, v_2\}$, $Y_i = \emptyset$ for $i > 0$.

(iv) $X_0 = \{v_0, v_1, v_2\}$, $X_1 = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$, $X_2 = \{v_0, v_1, v_2\}$, $X_i = \emptyset$ for $i > 2$ (so $|X_\bullet|$ is a 2-simplex); $Y_0 = \{v_0, v_1, v_2\}$, $Y_i = \emptyset$ for $i > 0$.

(v) $X_0 = \{v_0, v_1, v_2\}$, $X_1 = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$, $X_2 = \{v_0, v_1, v_2\}$, $X_i = \emptyset$ for $i > 2$ (so $|X_\bullet|$ is a 2-simplex); $Y_0 = \{v_0, v_1, v_2\}$, $Y_1 = \{\{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}\}$ (so $Y_\bullet$ is the "boundary" of $X_\bullet$), $Y_i = \emptyset$ for $i > 1$.

(6) Think about what would be involved in writing a computer program to compute simplicial homology, given a simplicial complex. You don't actually have to write anything for this problem, but you're invited to discuss it with me or Kristen.

(7) Read the exercises in chapter 2 of Lickorish. If any intrigue you, think about them. (Lickorish's exercises tend to be challenging.) Again, no need to write anything, but feel free to discuss ones that interest you with me or Kristen.

E-mail address: `rl2327@columbia.edu`