MATH W4052 PROBLEM SET 1
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(1) Cromwell Exercise 1.2. (You don't have to prove your answer to the last subquestion.)
(2) Cromwell Exercise 1.3.
(3) Let $f\colon S^1 \to \mathbb{R}^3$ be a continuous, injective map. Prove that the image $f(S^1)$ of f is homeomorphic to S^1. (Hint: this is easy.)
(4) Describe explicitly a smooth knot (image of a C^1 function with nonvanishing derivative) K in $\mathbb{R}^3$ so that the projection of K to $\mathbb{R}^2 = \{(x, y, 0)\} \subset \mathbb{R}^3$ has infinitely many transverse double-points (crossings at which the crossing strands are not tangent).

In the next few exercises, we give the beginning of another proof that any smooth knot has a knot diagram. The proof can be completed using similar ideas and a little more differential geometry. (We will use one key theorem, Sard's theorem, without proving it.)

(5) The Baire Category Theorem states: A (nonempty) complete metric space is not a countable union of nowhere-dense subsets. Prove this.

(Hint: Suppose X is a complete metric space and $X = \bigcup_{i=1}^{\infty} C_i$ where C_i is nowhere dense. We may assume C_i is closed (why?). Choose a point $p_1 \in X$ and an $\epsilon_1 > 0$ so that the ball $B(p_1, \epsilon_1)$ is contained in $X \setminus C_1$. Choose a point $p_2 \in B(p_1, \epsilon_1/2)$ and an $\epsilon_2 < \epsilon_1/2$ so that $B(p_2, \epsilon_2)$ is contained in $X \setminus (C_1 \cup C_2)$. Repeat.)
(6) Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 function. A point $p \in \mathbb{R}^n$ is a *critical point* of F if the total derivative $dF(p)$ of F is not surjective at p, i.e., if the matrix

$$\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial f_m/\partial x_1 & \partial f_m/\partial x_2 & \cdots & \partial f_m/\partial x_n \end{pmatrix},$$

where $f = (f_1, \dots, f_m)$, has rank less than m. A point $q \in \mathbb{R}^m$ is a *critical value* of F if $q = F(p)$ for some critical point p of F.

One version of Sard's theorem states: The set of critical values of a C^1 function F is a countable union of nowhere-dense sets. (Another version states that the set of critical values of F has measure 0.)

Use Sard's theorem to prove that there is no surjective, C^1 function $F\colon \mathbb{R}^1 \to \mathbb{R}^2$. (Hint: this is easy.)
(7) Use Sard's theorem to prove that there is no surjective, C^1 function $F\colon S^1 \to S^2$. (A function $F\colon S^1 \to S^2$ is differentiable if $(i \circ F \circ p)\colon \mathbb{R} \to \mathbb{R}^3$ is differentiable,

where $p\colon \mathbb{R} \to S^1$ is the map $\theta \mapsto e^{i\theta}$ and $i\colon S^2 \to \mathbb{R}^3$ is the usual inclusion of the unit sphere in $\mathbb{R}^3$.)

(8) Suppose $\gamma\colon S^1 \to \mathbb{R}^3$ is a *smooth embedding*, i.e., a C^∞ map with non-vanishing derivative. Prove that there is a plane P so that for the orthogonal projection map $\pi_P\colon \mathbb{R}^3 \to P$, $\pi_P \circ \gamma\colon S^1 \to P$ has non-vanishing derivative. (Hint: use the previous problem.)

Remark 1. Problem 3 shows, in particular, that the subspace of $\mathbb{R}^3$ illustrated in Cromwell's Figure 1.15 is homeomorphic to the standard circle. (This might seem surprising.)

Remark 2. In contrast to the result in Problem 6, there are continuous, surjective maps $\mathbb{R} \to \mathbb{R}^2$; such maps are called *space-filling curves*.

Remark 3. In the language of differential geometry, in Problem 8 you find a plane P so that $\pi_P \circ \gamma$ is an *immersion*.

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