MATH G4307 PROBLEM SET 4
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Exercises to turn in:

(E1) Hatcher 1.3.4 (p. 79)
(E2) Hatcher 1.3.10 (p. 79)
(E3) Hatcher 1.3.18 (p. 80)
(E4) Hatcher 1.3.23 (p. 81)
(E5) Given a group G, let $\mathrm{Hom}(G, \mathbb{Z})$ denote the set of group homomorphisms from G to $\mathbb{Z}$. Addition in $\mathbb{Z}$ (pointwise) makes $\mathrm{Hom}(G, \mathbb{Z})$ into an abelian group.

Recall that we defined $H^1(X) = [X, S^1]$.

Suppose that X is a path connected CW complex. Prove that $H^1(X) \cong \mathrm{Hom}(G, \mathbb{Z})$.

(Hint: there is a map $\Phi\colon H^1(X) \to \mathrm{Hom}(G, \mathbb{Z})$ defined by

$$(f\colon X \to S^1) \mapsto (f_*\colon \pi_1(X) \to \pi_1(S^1)).$$

Prove this is well-defined (basepoints!) and gives a group homomorphism. To prove injectivity, suppose $\Phi(f) = 0$ and lift f to the universal cover of S^1. To prove surjectivity, given a group homomorphism construct a corresponding map f starting with the 1-skeleton of X and then extending to the higher skeleta. Note that you can assume X has only one 0-cell, if you like, since both sides of the isomorphism are homotopy invariants.)

Problems to think about but not turn in:

(P1) In Problem (E5), if X is not a CW complex then many things can go wrong. Give examples where the isomorphism fails if:
  (a) X is not semi-locally simply connected. (You can use a subspace of $\mathbb{R}^2$.)
  (b) X is not locally path connected. (Again, you can use a subspace of $\mathbb{R}^2$, but you probably aren't in a position to prove your answer.)
  (c) X is not Hausdorff. (You can use a space with finitely many points.)
(P2) Let G be a graph, i.e., a 1-dimensional CW complex. Try to describe explicitly the universal cover of G, as a graph. Given such a description, extend the construction from class of the universal cover of a CW complex with a unique 0 cell to general connected CW complexes.
(P3) As mentioned in class, there is an analogy between (covering spaces and the fundamental group) and (field extensions and the Galois group). Try to make this analogy as explicit as you can.

(P4) Read through the remaining problems in this section, and do any that seem difficult, surprising or interesting. (Again, there are lots of very nice exercises in this section.)

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