MATH G4307 PROBLEM SET 10
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Version 2: Problem (E7) corrected.

Exercises to turn in:

(E1) Hatcher Exercise 3.2.1 (p. 228).

(E2) Hatcher Exercise 3.2.3 (p. 229).

(E3) Hatcher Exercise 3.2.7 (p. 229).

(E4) Hatcher Exercise 3.2.10 (p. 229). What's the point here, algebraically?

(E5) Hatcher Exercise 3.B.1 (p. 280).

(E6) Prove: Given basepoint-preserving maps $f\colon S^k \to S^k$, $g\colon S^\ell \to S^\ell$, the degree of $(f \wedge g)\colon (S^k \wedge S^\ell) \to (S^k \wedge S^\ell)$ is

$$\deg(f \wedge g) = \deg(f)\deg(g).$$

(We used this in class.)

(E7) A Steenrod square. Throughout this problem, you are welcome to work with $\mathbb{Z}/2$-coefficients. (I think, but do not promise, that the signs now work. Steenrod squares are typically operations on mod-2 cohomology, though this one seems to work over $\mathbb{Z}$.)

Consider a CW complex X. Let $\Delta\colon X \to X \times X$ denote the diagonal, let $\Delta_0\colon X \to X \times X$ be a cellular map homotopic to Δ, and let $T\colon X \times X \to X \times X$ be the map $T(x, y) = (y, x)$. Notice that $T \circ \Delta = \Delta$ but $T \circ \Delta_0 \neq \Delta_0$.

(a) Prove: there is a map $\Delta_1\colon C_n^{cell}(X) \to C_{n+1}^{cell}(X \times X)$ so that for each n-cell e of X,

$$T_* \circ (\Delta_0)_*(e) - (\Delta_0)_*(e) = \partial(\Delta_1 e) - \Delta_1(\partial e).$$

(Hint: Δ_0 is homotopic to $T \circ \Delta_0$.)

(b) Given cochains $a \in C^m(X)$, $b \in C^n(X)$ define $a \cup_1 b \in C_{cell}^{m+n-1}(X)$ by

$$(a \cup_1 b)(e) = (a \times b)(\Delta_1)_*(e),$$

or in other words

$$a \cup_1 b = \Delta_1^*(a \times b).$$

Prove that the coboundary of $a \cup_1 b$ is given by:

$$\delta(a \cup_1 b) = a \cup b - b \cup a + (\delta a) \cup_1 b + (-1)^{|a|} a \cup_1 (\delta b).$$

(c) Deduce that given a cocycle $a \in C^m(X)$, the element $\mathrm{Sq}^{m-1}(a) = a \cup_1 a \in C^{2m-1}(X)$ is a cocycle; and if a is a coboundary then $\mathrm{Sq}^{m-1}(a)$ is a coboundary.

So, Sq^{m-1} gives a map $H^m(X) \to H^{2m-1}(X)$. (We have not shown that Sq^{m-1} is independent of the choice of Δ_1.)

Remark. The operation Sq^{m-1} from Exercise (E7) is called a *Steenrod square*; and this is Steenrod's first definition of it. (By definition, $\mathrm{Sq}^m(a) = a \cup a$, for $a \in H^m(X)$.) A nice account can be found in Steenrod, Norman, "Cohomology operations, and obstructions to extending continuous functions." Advances in Mathematics 8 (1972) pp. 371–416. (There is also a short treatment in an appendix to Chapter 4 of Hatcher, and a more thorough exposition in Mosher and Tangora, *Cohomology Operations*.)

Problems to think about but not turn in:

(P1) Read Section 3.2 in Hatcher: we used a different approach in class, and it's important to understand both.

(P2) Recall that $C_*(X)$ is generated by maps $\sigma\colon \Delta^n \to X$. Given maps $\sigma_X\colon \Delta^n \to X$ and $\sigma_Y\colon \Delta^n \to Y$ there's an obvious map $\sigma_X \times \sigma_Y\colon \Delta^n \to X \times Y$. Moreover, every map $\Delta^n \to X \times Y$ has this form. Why doesn't this prove that $H_n(X \times Y) = H_n(X) \otimes H_n(Y)$? (Note that this would be a very different result from the Künneth theorem—and false.)

(P3) The proof of the cellular approximation theorem uses the following: given a map $f\colon S^n \to S^m$ with $m > n$, f is homotopic to a map which is not surjective. Hatcher gives a proof in the style of piecewise-linear approximation, in Chapter 4. You can also prove this using smooth techniques:
   (a) Show that if $f\colon S^n \to S^m$ $(m > n)$ is a smooth map then f is not surjective. (Hint: look up Sard's theorem.)
   (b) Show that any continuous map $f\colon S^n \to \mathbb{R}^{m+1}$ can be approximated arbitrarily well (in the C^0 norm) by a smooth map, by convolving with an (appropriate) approximation to the identity (i.e., a C^∞ bump function of total area 1, vanishing outside the ball of radius ϵ around the origin).
   (c) Use the previous part to show that any map $f\colon S^n \to S^m$ is homotopic to a smooth map $S^n \to S^m$.

(P4) Read through the remaining problems in these sections, and do any that seem difficult, surprising or interesting. (There are lots of very nice exercises, especially in Section 3.2.)

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