MATH W4051 PROBLEM SET 2
DUE SEPTEMBER 22, 2009.

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(1) Let $\mathcal{C}^\infty(\mathbb{R})$ denote the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that f is differentiable to all orders (i.e., $f^{(n)}$ exists for all $n \geq 0$). Notice that $\mathcal{C}^\infty(\mathbb{R})$ is a vector space in an obvious way.

We endow $\mathcal{C}^\infty(\mathbb{R})$ with two different metrics. Let[1]

$$d_0(f,g) = \sup\{|f(x) - g(x)| \mid x \in \mathbb{R}\}$$
$$d_1(f,g) = d_0(f,g) + \sup\{|f'(x) - g'(x)| \mid x \in \mathbb{R}\}$$

(d_0 is called the $\mathcal{C}^0$-metric and d_1 is called the $\mathcal{C}^1$-metric.)

(a) Convince yourself that d_0 and d_1 are, in fact, metrics. (You don't have to write anything for this part.)

(b) Is the topology induced by d_0 finer or coarser than the topology induced by d_1?

(c) Define a map $D\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ by $D(f)(x) = f'(x)$. Prove that D gives a continuous map $(\mathcal{C}^\infty, d_1) \to (\mathcal{C}^\infty, d_0)$.

(d) Prove that D does *not* give a continuous map $(\mathcal{C}^\infty, d_0) \to (\mathcal{C}^\infty, d_0)$. (If you haven't seen this before, this should surprise you: the map D is linear but not necessarily continuous!)

(2) (a) Let X be a set and $\mathcal{B}$ a sub-basis for a topology on X. Then the topology generated by $\mathcal{B}$ is the coarsest topology on X such that every set in $\mathcal{B}$ is open. Formulate precisely what this means.

(b) Prove it.

(c) For Y and Z topological spaces, the product topology on $Y \times Z$ is the finest topology on $Y \times Z$ such that for any topological space X and continuous maps $f\colon X \to Y$, $g\colon X \to Z$, $(f,g)\colon X \to Y \times Z$ is continuous. Prove this.

The product topology is also the coarsest topology so that the projections $\pi_Y\colon Y \times Z \to Y$ and $\pi_Z\colon Y \times Z \to Z$ are continuous. Prove this, too.

(d) Analogous statements hold for arbitrary (possibly infinite) products. Formulate and prove them.

(This problem should make you feel lucky that the product topology exists: it's the finest topology with one property you want, but the coarsest with another, so it's the only topology with both.)

(3) Munkres 17.10

(4) Munkres 17.16

(5) Munkres 18.6

(6) Munkres 19.6

(7) Read Section 3 of the "Ordinals" handout (or post-out?) and do problems 3 and 4.

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[1] for a set S of real numbers, recall that $\sup(S)$ is the supremum or least upper bound of S, i.e., the smallest real number r such that for all $s \in S$, $s \leq r$.