MATH W4051 PROBLEM SET 1
DUE SEPTEMBER 15, 2009.

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(1) Use the ϵ-δ definition of continuity to prove that the function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous.[1]

(2) Given metric spaces (X, d_X) and (Y, d_Y), we define the *product metric* $d_{X\times Y}$ on $X\times Y$ by setting
$$d_{X\times Y}\left((x,y),(x',y')\right) = \sqrt{d_X(x,x')^2 + d_Y(y,y')^2}.$$
  (a) Prove that $(X \times Y, d_{X\times Y})$ is, in fact, a metric space.
  (b) Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces, and $f\colon X \to Y$, $g\colon X \to Z$ continuous maps. Prove that the map $(f, g)\colon X \to Y \times Z$ is continuous (where $Y \times Z$ is given the product metric).

(3) Given metric spaces (X, d_X) and (Y, d_Y), a map $f\colon X \to Y$ is called an *isometry* if
  - f is surjective and
  - for all $p, q \in X$, $d_Y(f(p), f(q)) = d_X(p, q)$.

  Spaces X and Y are called isometric if there exists an isometry $f\colon X \to Y$.
  (a) Prove that isometries are necessarily injective.
  (b) Show that if X and Y are isometric then X and Y are homeomorphic.
  (c) Give an example of two (metric) spaces which are homeomorphic but not isometric. (Hint: see Problem (5).)

(4) Homeomorphisms preserve topological properties. As an example of this, prove: Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be homeomorphic topological spaces. Then $(X, \mathcal{T})$ is metrizable if and only if $(Y, \mathcal{S})$ is metrizable.

(5) Let (X, d_X) be a metric space and $r \in \mathbb{R}$. We say X has diameter at most r if for any $x, y \in X$, $d_X(x, y) \le r$. We say X has *finite diameter* if X has diameter at most r for some $r \in \mathbb{R}$.
  (a) Give an example of a metric space with diameter at most 1, and a metric space which does not have finite diameter.
  (b) Prove that having diameter at most r is a metric property. That is, prove that if (X, d_X) is isometric to (Y, d_Y) and X has diameter at most r then Y has diameter at most r.
  (c) Prove that having diameter at most r is *not* a topological property. That is, find metric spaces (X, d_X) and (Y, d_Y) such that X is homeomorphic to Y, X has diameter at most r and Y does not have diameter at most r.
  (d) Prove that having finite diameter is not a topological property.

(6) Munkres problem 13.5. (If you find this confusing, do 13.4 first.)

(7) (Countability) Read Munkres Section 7. Do problem 7.1. (If this is all new to you, also do problem 7.4.)

(8) Read Sections 1 and 2 of the "Ordinals" handout and do Problem 1 in it. (*Hint:* use induction.)

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[1] If you find this hard, do several more, like $f(x) = x^3\colon \mathbb{R} \to \mathbb{R}$, $f(x, y) = x^2 + y^2\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x) = 1/x\colon (0, 1) \to (1, \infty)$, etc., until they become fairly easy.