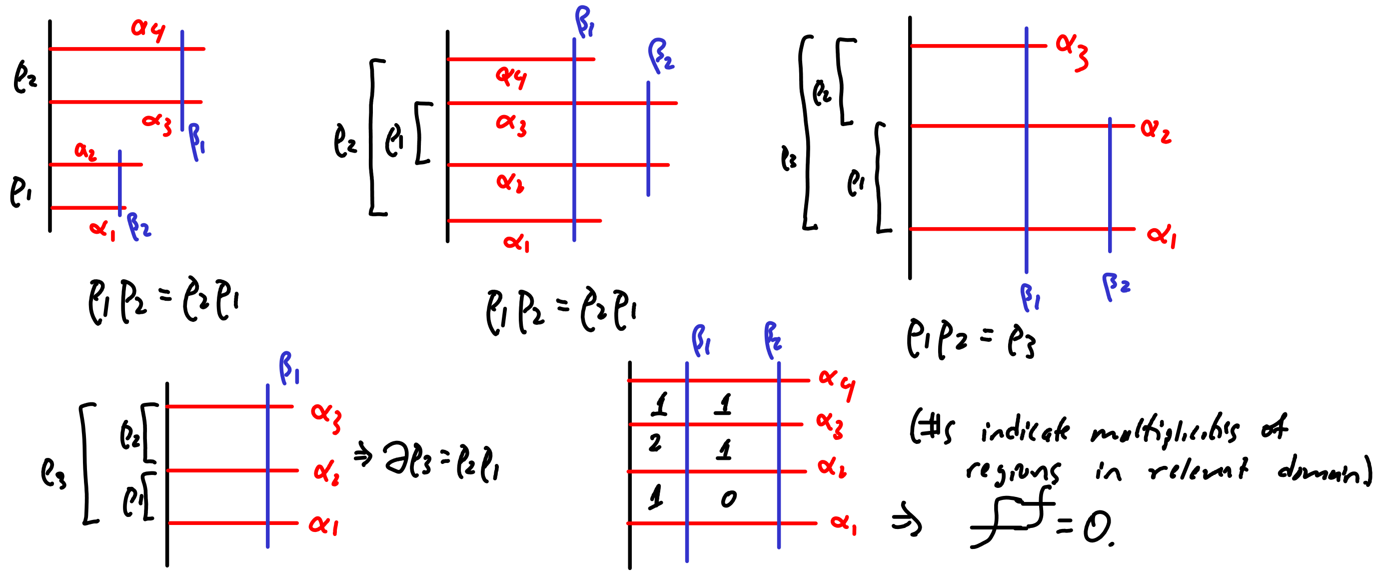
BORDERED FLOER HOMOLOGY HOMEWORK 2

ROBERT LIPSHITZ

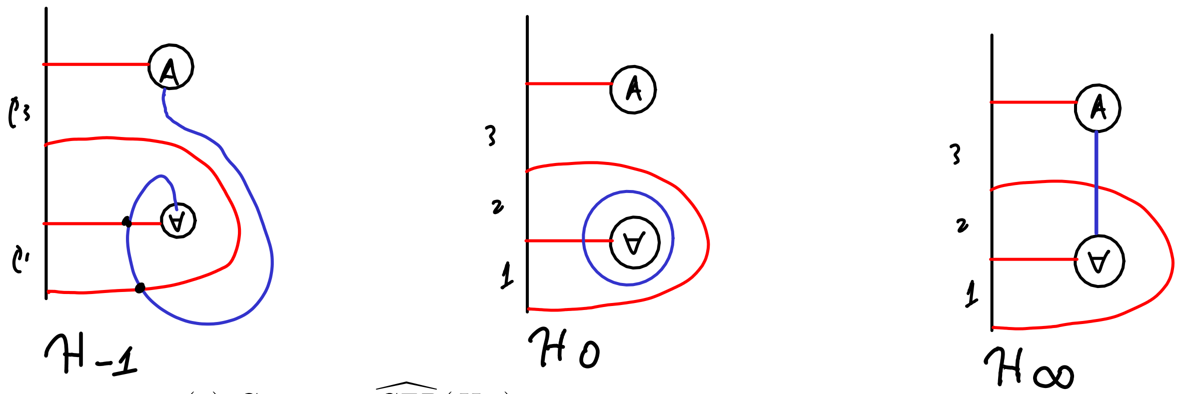
(1) Let $a(\rho)$ denote the algebra element associated to a chord ρ. Suppose that $\mathcal{A}(\mathcal{Z})$ is generated by chords, and that the module ${}_{\mathcal{A}(\mathcal{Z})}\widehat{CFD}(\mathcal{H})$ is defined as in the lecture. Verify that the diagram fragments shown induce the specified relations on $\mathcal{A}$. (See [4] for details.)



(2) Recall that a strongly bordered 3-manifold with two boundary components consists of a Y^3 with $\partial Y = \partial_L Y \amalg \partial_R Y$; parameterizations of $\partial_L Y$ and $\partial_R Y$ by surfaces $F(\mathcal{Z}_L)$ and $F(\mathcal{Z}_R)$; and a framed arc γ from $\partial_L Y$ to $\partial_R Y$ (connecting the basepoints z in $\partial_L Y$ and $\partial_R Y$, and with framing pointing into the 2-handle of $\partial_L Y$ and $\partial_R Y$). Define Heegaard diagrams for strongly bordered 3-manifolds with two boundary components, in analogy to bordered Heegaard diagrams for bordered 3-manifolds with connected boundary. (See [5, Section 5.1] for details.)

(3) Below are Heegaard diagrams for the 0-framed, -1-framed and ∞-framed solid tori. We computed the invariant $\widehat{CFD}(H_0)$ and $\widehat{CFD}(H_{-1})$ in lecture.

Date: July 20, 2010.



(a) Compute $\widehat{CFD}(H_\infty)$.

(b) Write down an exact sequence

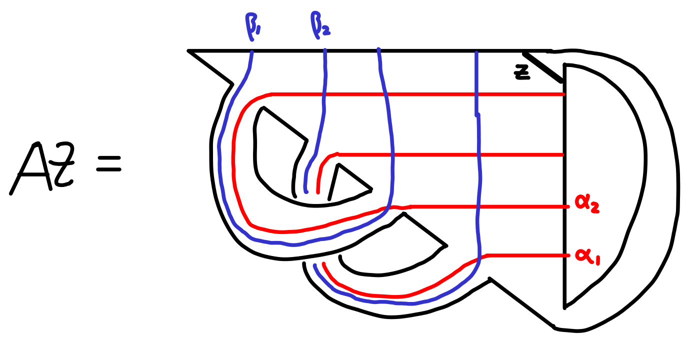
$$0 \to \widehat{CFD}(H_\infty) \to \widehat{CFD}(H_{-1}) \to \widehat{CFD}(H_0) \to 0.$$

What does this imply about $\widehat{HF}(Y)$?

(4) This exercise relates to a diagram from [1] which can be used to show that

$$\widehat{HF}(-Y_1 \cup_F Y_2) \cong \mathrm{Ext}_{\mathcal{A}(-F)}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2)) \cong \mathrm{Ext}_{\mathcal{A}(F)}(\widehat{CFA}(Y_1), \widehat{CFA}(Y_2)).$$

(See [1] or [2].) For notational convenience, we will work in the genus 1 case. Let AZ denote the following strongly bordered Heegaard diagram:



(Unlike the diagrams we will use in lecture, this one has α-arcs meeting one boundary component and β-arcs meeting the other.)

(a) Prove (by direct computation) that $\widehat{CFAA}(\mathsf{AZ})$ is isomorphic to $\mathcal{A}(T^2)$, as an $\mathcal{A}(T^2)$-bimodule. (Hint: AZ is a *nice diagram*; see [3, Section 8], particularly Proposition 8.4.)

(b) Aside: Use the pairing theorem and the previous part to show that if $\mathbb{I}_{\mathcal{Z}}$ denotes the identity Heegaard diagram for $\mathcal{Z}$ then $\widehat{CFDA}(\mathbb{I}) \simeq \mathcal{A}$ as an $\mathcal{A}$-bimodule.

(c) Given a Heegaard diagram $\mathcal{H}$, let $\overline{\mathcal{H}}$ be the orientation-reverse of $\mathcal{H}$. Prove that $\widehat{CFD}(\overline{\mathcal{H}}) = \widehat{CFD}(\mathcal{H})^*$, the dual of $\widehat{CFD}(\mathcal{H})$.

(d) Suppose $\mathcal{H}$ is a bordered Heegaard diagram for a 3-manifold Y with connected boundary. Let $\mathcal{H}^\beta$ denote the result of relabeling the α-curves in $\mathcal{H}$ as β-curves and the β-curves as α-curves. Show that $\mathcal{H}^\beta \cup_\partial \mathsf{AZ}$ is a bordered Heegaard diagram for $-Y$.

(e) Challenge: Let $\mathcal{H}$ denote the result of gluing two copies of AZ along the boundary component intersecting the β-arcs (so $\mathcal{H}$ is an α-α-bordered Heegaard diagram.) What strongly bordered 3-manifold does $\mathcal{H}$ represent?

References

[1] Denis Auroux, *Fukaya categories of symmetric products and bordered Heegaard-Floer homology*, 2010, arXiv:1001.4323.

[2] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston, *Heegaard Floer homology as morphism spaces*, in preparation.

[3] ———, *Bordered Heegaard Floer homology: Invariance and pairing*, 2008, arXiv:0810.0687.

[4] ———, *Slicing planar grid diagrams: A gentle introduction to bordered Heegaard Floer homology*, 2008, arXiv:0810.0695.

[5] ———, *Bimodules in bordered Heegaard Floer homology*, 2010, arXiv:1003.0598.

Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: lipshitz@math.columbia.edu