Bordered Heegaard Floer homology

R. Lipshitz, P. Ozsváth and D. Thurston

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1. Review of Heegaard Floer
2. Basic properties of bordered *HF*
3. Bordered Heegaard diagrams
4. The algebra
5. The cylindrical setting for Heegaard Floer
6. The module $\widehat{CFD}$
7. The module $\widehat{CFA}$
8. The pairing theorem
9. Four-dimensional information from bordered *HF*.

Classical Heegaard Floer theory assigns...

To Y^3 closed, oriented	chain complexes $\widehat{CF}(Y)$, $CF^+(Y)$, ... well-defined up to homotopy equivalence.
To $W^4 \colon Y_1^3 \to Y_2^3$ smooth, oriented	chain maps $\hat{F}_W \colon \widehat{CF}(Y_1) \to \widehat{CF}(Y_2)$,... well-defined up to chain homotopy.

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Theorem

If $W_1\colon Y_1 \to Y_2$ and $W_2\colon Y_2 \to Y_3$ then $\hat{F}_{W_1 \cup_{Y_2} W_2} = \hat{F}_{W_2} \circ \hat{F}_{W_1}$, ...

(I'm omitting spin^c-structures)

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It's like having only de Rham cohomology, except via nonlinear equations and without the Mayer-Vietoris theorem.

Bordered Floer homology

- The rest of the talk is about joint work with Peter Ozsváth and Dylan Thurston.
- Most of it can be found in "Bordered Heegaard Floer homology: Invariance and pairing," `arXiv:0810.0687`. (It's quite long.)
- We also wrote an expository paper about some of the ideas, "Slicing planar grid diagrams: a gentle introduction to bordered Heegaard Floer homology," `arXiv:0810.0695`, which we hope is easy to read.

The goals of bordered Floer homology

Theorem

(Ozsváth-Szabó) If $Y = Y_1 \# Y_2$ *then*
$\widehat{CF}(Y) \cong \widehat{CF}(Y_1) \otimes_{\mathbb{F}_2} \widehat{CF}(Y_2)$.

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Bordered Floer theory extends this more general decompositions of 3-manifolds along surfaces.

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such that

- If $Y = Y_1 \cup_F Y_2$ then

$$\widehat{CF}(Y) = \widehat{CFA}(Y_1) \otimes_{\mathcal{A}(F)} \widehat{CFD}(Y_2).$$

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Bordered Y^3, $\partial Y^3 = F$	compact, oriented 3-manifold with connected boundary, orientation-preserving homeomorphism $F \to \partial Y$	Right A_∞-module $\widehat{CFA}(Y)$ over $\mathcal{A}(F)$, Left *dg*-module $\widehat{CFD}(Y)$ over $\mathcal{A}(-F)$, well-defined up to homotopy equiv.

Satisfying the pairing theorem:

Theorem

If $\partial Y_1 = F = -\partial Y_2$ *then*

$$\widehat{CF}(Y_1 \cup_\partial Y_2) \simeq \widehat{CFA}(Y_1)\widetilde{\otimes}_{\mathcal{A}(F)}\widehat{CFD}(Y_2).$$

Further structure (in progress):

- To an $\phi \in \mathrm{MCG}(F)$, bimodules $\widehat{CFDA}(\phi)$, $\widehat{CFDA}(\phi)$.

$$\widehat{CFA}(\phi(Y)) \simeq \widehat{CFA}(Y) \,\widetilde{\otimes}_{\mathcal{A}(F)}\, \widehat{CFDA}(\phi)$$
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- To F, bimodules $\widehat{CFDD}$ and $\widehat{CFAA}$, such that

$$\widehat{CFD}(Y) \simeq \widehat{CFA}(Y) \,\widetilde{\otimes}_{\mathcal{A}(F)}\, \widehat{CFDD}$$
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Theorem

Suppose $CFK^-(K) \simeq CFK^-(K')$. Let K_C (resp. K'_C) be the satellite of K (resp. K') with companion C. Then $HFK^-(K_C) \cong HFK^-(K'_C)$.

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- It's good for computations:
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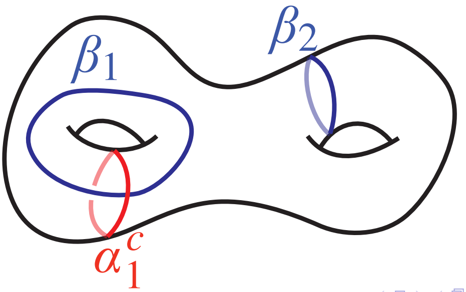
  - In fact, you can compute $\hat{F}_W$ for any W^4.

Bordered Heegaard diagrams

- Let $(\overline{\Sigma}_g, \alpha_1^c, \dots, \alpha_{g-k}^c, \beta_1, \dots, \beta_g)$ be a Heegaard diagram for a Y^3 with bdy.



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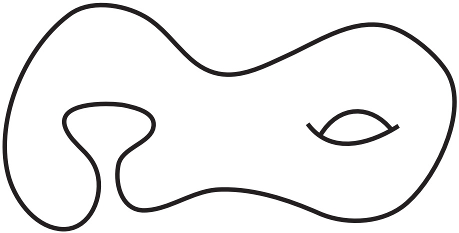

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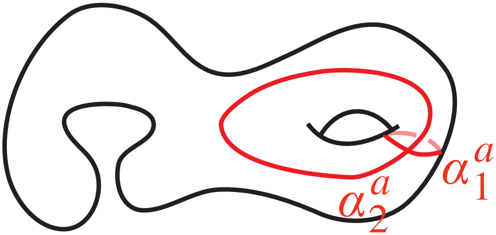

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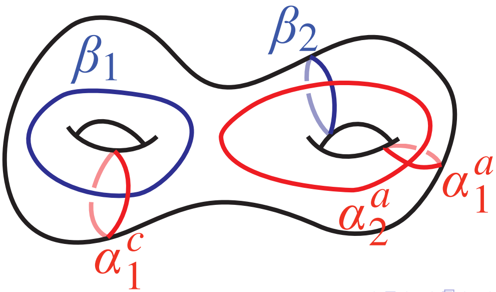
- These give circles $\alpha_1^a, \dots, \alpha_{2k}^a$ in $\overline{\Sigma}$.



- Let $\Sigma = \overline{\Sigma} \setminus \mathbb{D}_\epsilon(p)$.
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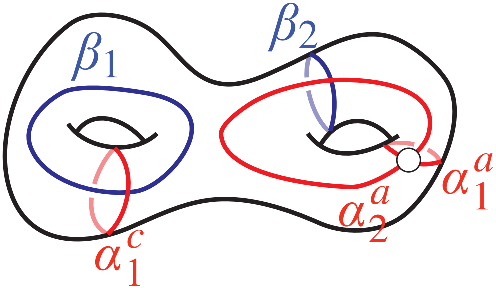

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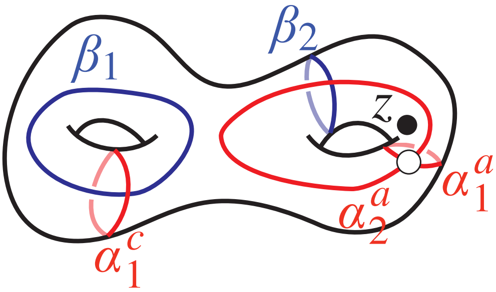
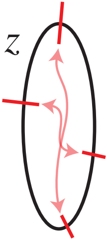
- Fix also $z \in \overline{\Sigma}$ near p.



A small circle near p looks like:



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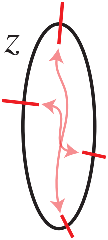
This is called a *pointed matched circle* $\mathcal{Z}$.



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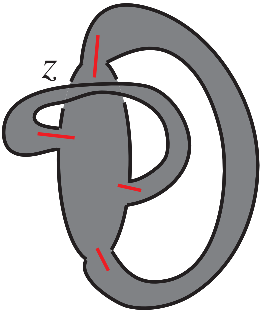

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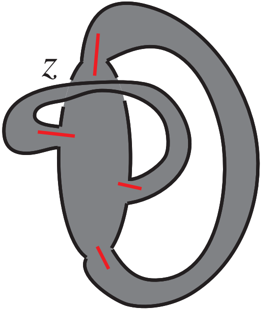
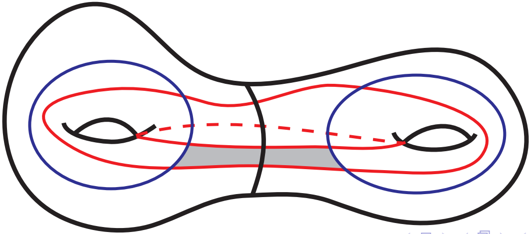
We will associate a *dg* algebra $\mathcal{A}(\mathcal{Z})$ to $\mathcal{Z}$.



Where the algebra comes from.

- Decomposing ordinary $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ into bordered H.D.'s $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1) \cup (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$, would want to consider holomorphic curves crossing $\partial\Sigma_1 = \partial\Sigma_2$.



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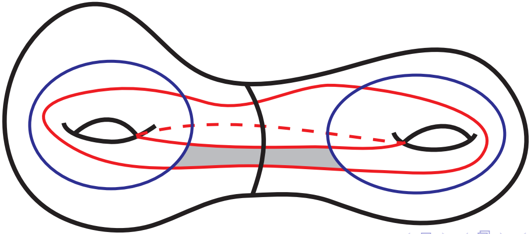

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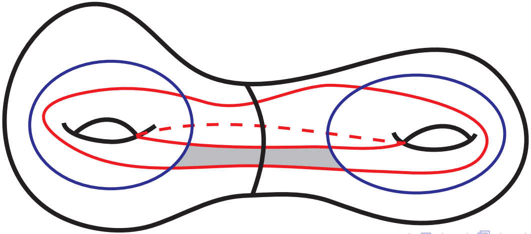
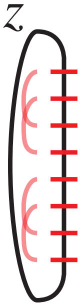
- This suggests the algebra should have to do with Reeb chords in $\partial\Sigma_1$ relative to $\boldsymbol{\alpha} \cap \partial\Sigma_1$.
- Analyzing some simple models, in terms of *planar grid diagrams*, suggested the product and relations in the algebra.



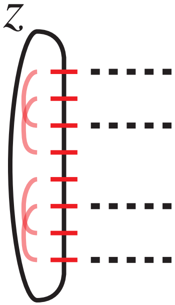
So...

- Let $\mathcal{Z}$ be a pointed matched circle, for a genus k surface.



So...

- Let $\mathcal{Z}$ be a pointed matched circle, for a genus k surface.
- Primitive idempotents of $\mathcal{A}(\mathcal{Z})$ correspond to k-element subsets I of the $2k$ pairs in $\mathcal{Z}$.
- We draw them like this:



- A pair (I, ρ), where ρ is a Reeb chord in $\mathcal{Z} \setminus z$ starting at I specifies an algebra element $a(I, \rho)$.
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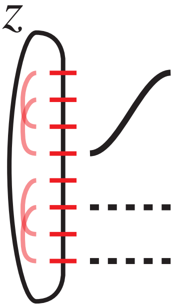

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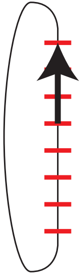

More generally, given $(I, \boldsymbol{\rho})$ where $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_\ell\}$ is a set of Reeb chords starting at I, with:

- $i \neq j$ implies ρ_i and ρ_j start and end on different pairs.
- $\{\text{starting points of } \rho_i\text{'s}\} \subset I$.

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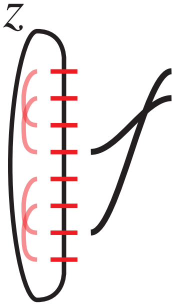

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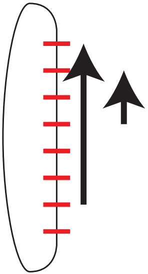
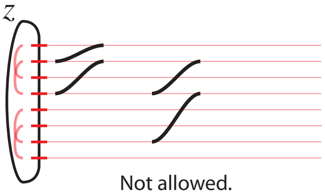


These generate $\mathcal{A}(\mathcal{Z})$ over $\mathbb{F}_2$.

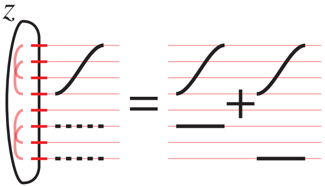
That is, $\mathcal{A}(\mathcal{Z})$ is the subalgebra of the algebra of *k-strand, upward-veering flattened braids on* $4k$ *positions* where:

- no two start or end on the same pair



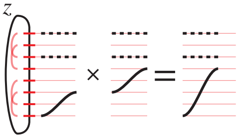
Not allowed.

- Algebra elements are fixed by "horizontal line swapping".



Multiplication...

...is concatenation if sensible, and zero otherwise.



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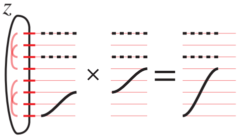

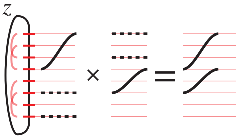

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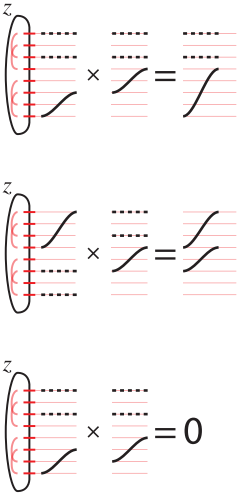
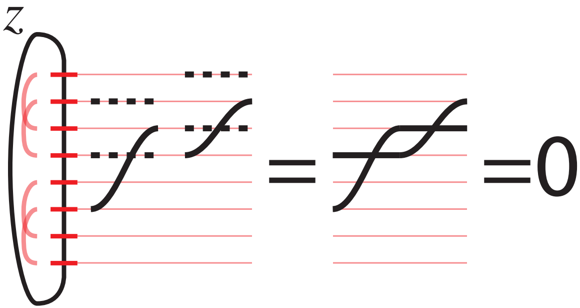


Double crossings

We impose the relation

$$(\text{double crossing}) = 0.$$

e.g.,

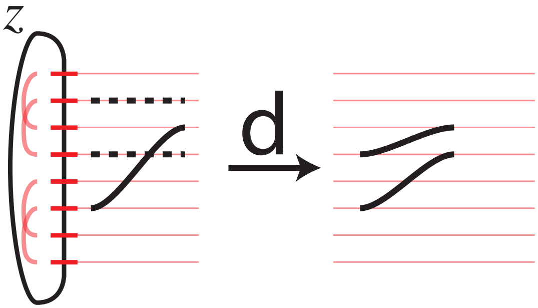


The differential

There is a differential d by

$$d(a) = \sum \text{smooth one crossing of } a.$$

e.g.,



Why?

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Studying degenerations of holomorphic curves.

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Studying degenerations of holomorphic curves.

They can all be deduced from some simple examples.
See `arXiv:0810.0695`.

Algebra – summary

- The algebra is generated by the Reeb chords in $\mathcal{Z}$, with certain relations. e.g.,

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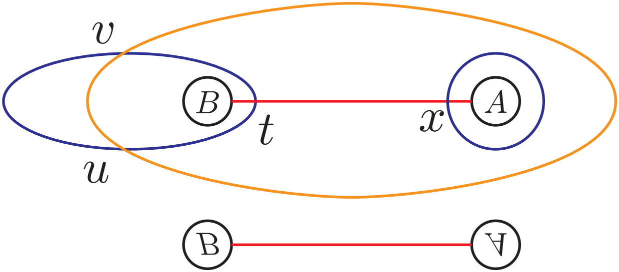
- The algebra is generated by the Reeb chords in $\mathcal{Z}$, with certain relations. e.g.,
  - Multiplying consecutive Reeb chords concatenates them.
  - Far apart Reeb chords commute.
- The algebra is finite-dimensional over $\mathbb{F}_2$, and has a nice description in terms of flattened braids.

The cylindrical setting for classical $\widehat{CF}$:

Fix an ordinary H.D. $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. (Here, $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$.)

- The chain complex $\widehat{CF}$ is generated over $\mathbb{F}_2$ by g-tuples $\{x_i \in \alpha_{\sigma(i)} \cap \beta_i\} \subset \boldsymbol{\alpha} \cap \boldsymbol{\beta}$. ($\sigma \in S_g$ is a permutation.) (cf. $T_\alpha \cap T_\beta \subset Sym^g(\Sigma)$.)



Generators: $\{u, x\}, \{v, x\}$.

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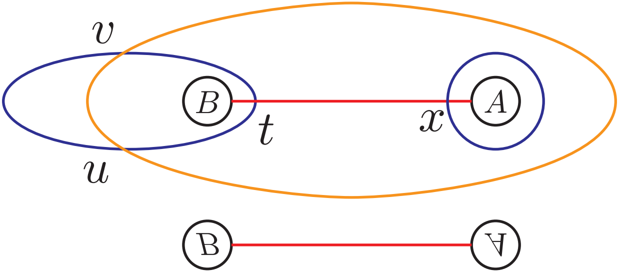
- The differential counts embedded holomorphic maps

$$(S, \partial S) \to (\Sigma \times [0,1] \times \mathbb{R}, (\boldsymbol{\alpha} \times 1 \times \mathbb{R}) \cup (\boldsymbol{\beta} \times 0 \times \mathbb{R}))$$

  asymptotic to $\mathbf{x} \times [0,1]$ at $-\infty$ and $\mathbf{y} \times [0,1]$ at $+\infty$.
- For $\widehat{CF}$, curves may not intersect $\{z\} \times [0,1] \times \mathbb{R}$.

Example of $\widehat{CF}$



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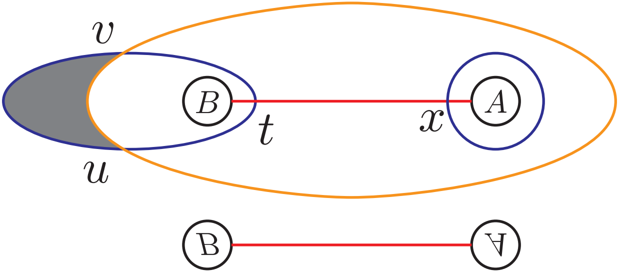

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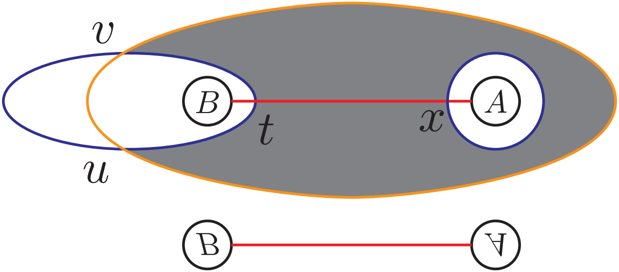

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- The $e\infty$ asymptotics are *Reeb chords* $\rho_i \times (1, t_i)$.
- The asymptotics $\rho_{i_1}, \dots, \rho_{i_\ell}$ of u inherit a partial order, by $\mathbb{R}$-coordinate.

Generators of $\widehat{CFD}$...

Fix a bordered Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$

$\widehat{CFD}(\Sigma)$ is generated by g-tuples $\mathbf{x} = \{x_i\}$ with:

- one x_i on each β-circle
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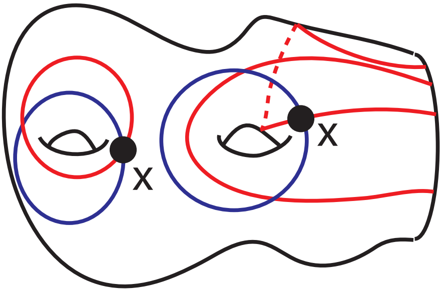

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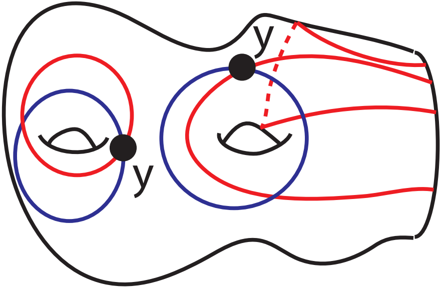
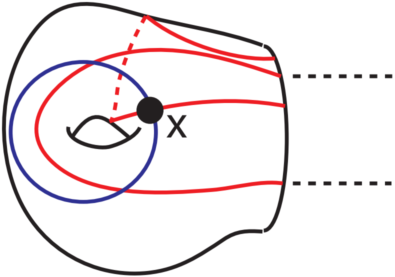


...and associated idempotents.

- To $\mathbf{x}$, associate the idempotent $I(\mathbf{x})$, the α-arcs **not** occupied by $\mathbf{x}$.



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- So, if I is a primitive idempotent, $I\mathbf{x} = 0$ if $I \neq I(\mathbf{x})$ and $I(\mathbf{x})\mathbf{x} = \mathbf{x}$.

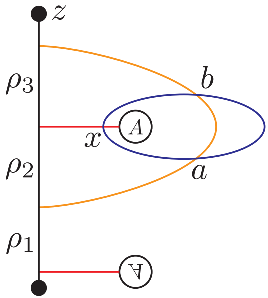
The differential on $\widehat{CFD}$.

$$d(\mathbf{x}) = \sum_{\mathbf{y}} \sum_{(\rho_1,\dots,\rho_n)} (\#\mathcal{M}(\mathbf{x},\mathbf{y};\rho_1,\dots,\rho_n))\, a(\rho_1, I(\mathbf{x}))\cdots a(\rho_n, I_n)\mathbf{y}.$$

where $\mathcal{M}(\mathbf{x},\mathbf{y};\rho_1,\dots,\rho_n)$ consists of holomorphic curves asymptotic to

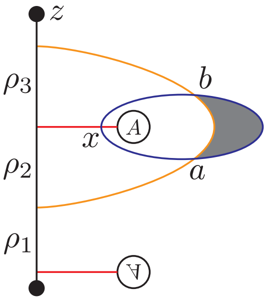
- $\mathbf{x}$ at $-\infty$
- $\mathbf{y}$ at $+\infty$
- $\rho_1,\dots,\rho_n$ at $e\infty$.

Example D1: a solid torus.



$$
\begin{aligned}
d(b) &= a + \rho_3 x \\
d(x) &= \rho_2 a \\
d(a) &= 0.
\end{aligned}
$$

Example D1: a solid torus.



$$d(b) = \mathbf{a} + \rho_3 x$$
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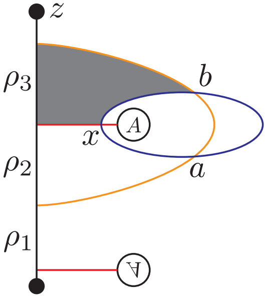

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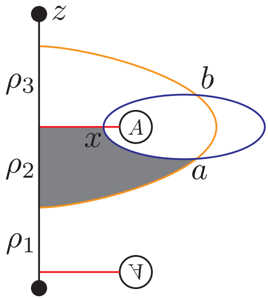

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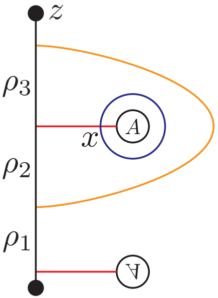

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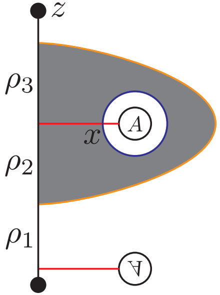

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Comparison of the two examples.

First chain complex:

$$\begin{tikzcd} a \arrow[d, "\rho_3"'] \arrow[dr, "1"] & \\ x \arrow[r, "\rho_2"] & b \end{tikzcd}$$

Second chain complex:

$$x \xrightarrow{\rho_{23}} x$$

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They're homotopy equivalent. In fact:

Theorem

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', z')$ are pointed bordered Heegaard diagrams for the same bordered Y^3 then $\widehat{CFD}(\Sigma)$ is homotopy equivalent to $\widehat{CFD}(\Sigma')$.

Generators and idempotents of $\widehat{CFA}$.

Fix a bordered Heegaard diagram $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$
$\widehat{CFA}(\Sigma)$ is generated by the same set as $\widehat{CFD}$: g-tuples $\mathbf{x} = \{x_i\}$ with:

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This is much smaller than $\widehat{CFD}$.

The differential on $\widehat{CFA}$...

...counts only holomorphic curves contained in a compact subset of Σ, i.e., with no asymptotics at $e\infty$.

The module structure on $\widehat{CFA}$

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- Given a set $\boldsymbol{\rho}$ of Reeb chords, define

$$\mathbf{x} \cdot a(J(\mathbf{x}), \boldsymbol{\rho}) = \sum_{\mathbf{y}} \left(\#\mathcal{M}(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})\right) \mathbf{y}$$

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  - $\mathbf{x}$ at $-\infty$.
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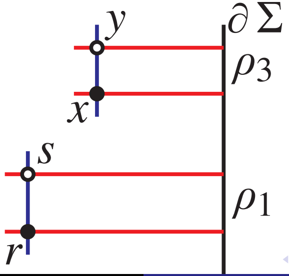
  - $\boldsymbol{\rho}$ at $e\infty$, *all at the same height.*

A local example of the module structure on $\widehat{CFA}$.

- Consider the following piece of a Heegaard diagram, with generators $\{r, x\}, \{s, x\}, \{r, y\}, \{s, y\}$.



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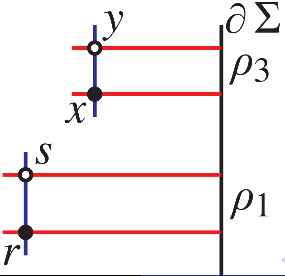

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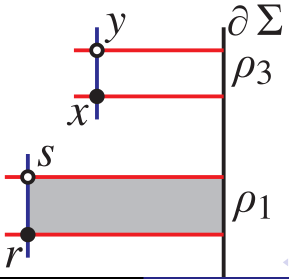

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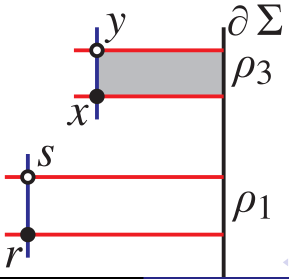

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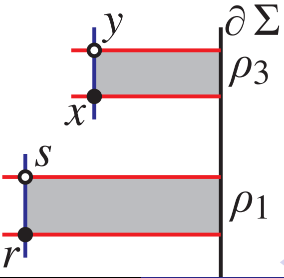
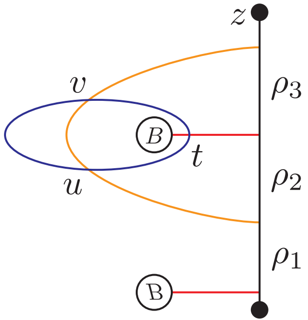


Example A1: a solid torus.



$$d(u) = v$$
$$u\rho_2 = t$$
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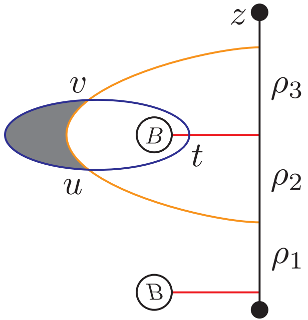

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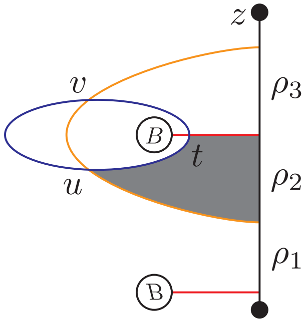

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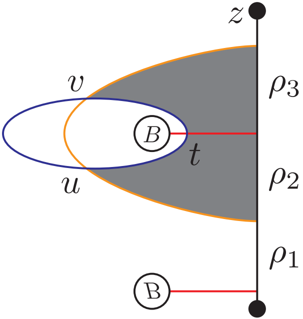

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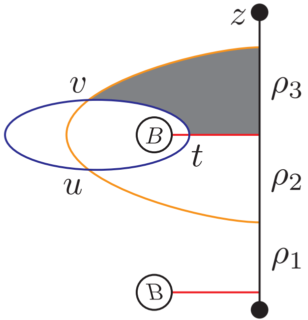

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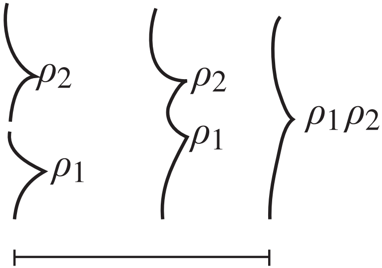
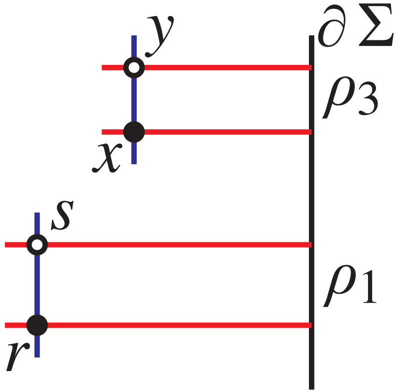
$$\mathbf{t}\rho_3 = \mathbf{v} \; .$$

Why associativity should hold...

- $(\mathbf{x} \cdot \rho_i) \cdot \rho_j$ counts curves with ρ_i and ρ_j infinitely far apart.
- $\mathbf{x} \cdot (\rho_i \cdot \rho_j)$ counts curves with ρ_i and ρ_j at the same height.
- These are ends of a 1-dimensional moduli space, with height between ρ_i and ρ_j varying.

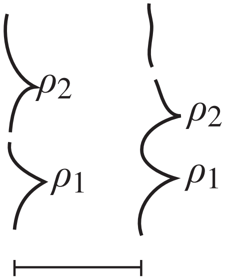


The local model again.



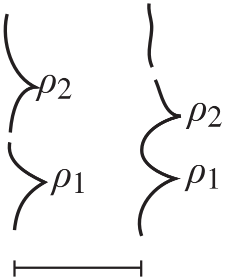
...and why it doesn't.

- But this moduli space might have other ends: broken flows with ρ_1 and ρ_2 at a fixed nonzero height.



...and why it doesn't.

- But this moduli space might have other ends: broken flows with ρ_1 and ρ_2 at a fixed nonzero height.
- These moduli spaces – $\mathcal{M}(\mathbf{x},\mathbf{y};(\rho_1,\rho_2))$ – measure failure of associativity. So...



Higher A_∞-operations

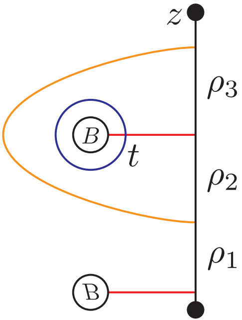
Define

$$m_{n+1}(\mathbf{x}, a(\boldsymbol{\rho}_1), \dots, a(\boldsymbol{\rho}_n)) = \sum_{\mathbf{y}} (\#\mathcal{M}(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n)))\, \mathbf{y}$$

where $\mathcal{M}(\mathbf{x}, \mathbf{y}; (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_n))$ consists of holomorphic curves asymptotic to

- $\mathbf{x}$ at $-\infty$.
- $\mathbf{y}$ at $+\infty$.
- $\boldsymbol{\rho}_1$ all at one height at $e\infty$, $\boldsymbol{\rho}_2$ at some other (higher) height at $e\infty$, and so on.

Example A2: same torus, different diagram.



$$m_3(x, \rho_3, \rho_2) = x$$
$$m_4(x, \rho_3, \rho_{23}, \rho_2) = x$$
$$m_5(x, \rho_3, \rho_{23}, \rho_{23}, \rho_2) = x$$
$$\vdots$$

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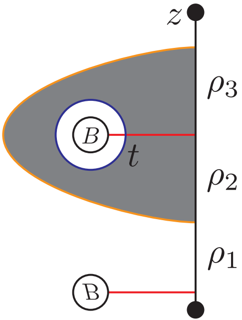

$$\mathbf{m_3(x, \rho_3, \rho_2) = x}$$
$$m_4(x, \rho_3, \rho_{23}, \rho_2) = x$$
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$$\vdots$$

Comparison of the two examples.

First chain complex:

$$\begin{tikzcd} u \arrow[d, "m_2(\cdot,\rho_2)"'] \arrow[dr, "1+\rho_{23}"] & \\ x \arrow[r, "m_2(\cdot,\rho_3)"] & v \end{tikzcd}$$

Second chain complex:

$$x \xrightarrow{m_3(\cdot,\rho_3,\rho_2)+m_4(\cdot,\rho_3,\rho_{23},\rho_2)+\dots} x$$

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They're A_∞ homotopy equivalent (exercise).
Suggestive remark:

$$(1+\rho_{23})^{-1} \text{“=”} 1+\rho_{23}+\rho_{23},\rho_{23}+\dots$$
$$\rho_3(1+\rho_{23})^{-1}\rho_2 \text{“=”} \rho_3,\rho_2+\rho_3,\rho_{23},\rho_2+\dots.$$

In general:

Theorem

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ *and* $(\Sigma, \boldsymbol{\alpha}', \beta', z')$ *are pointed bordered Heegaard diagrams for the same bordered* Y^3 *then* $\widehat{CFA}(\Sigma)$ *is* A_∞*-homotopy equivalent to* $\widehat{CFA}(\Sigma')$.

The pairing theorem

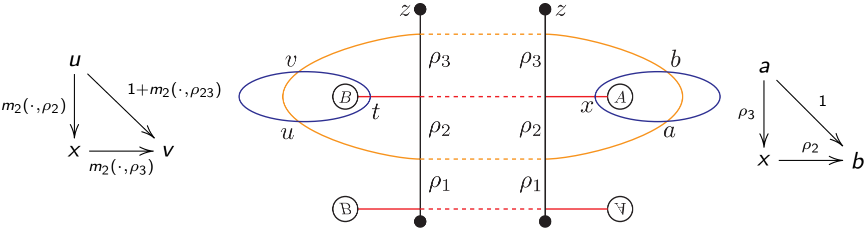
Recall:

Theorem

If $\partial Y_1 = F = -\partial Y_2$ *then*

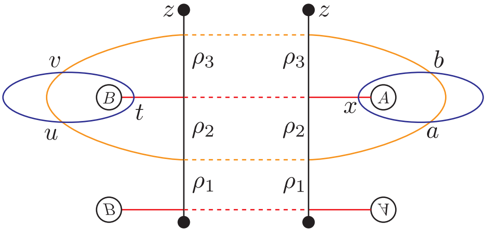
$$\widehat{CF}(Y_1 \cup_\partial Y_2) \simeq \widehat{CFA}(Y_1) \widetilde{\otimes}_{\mathcal{A}(F)} \widehat{CFD}(Y_2).$$

We'll illustrate this with three examples.



Generators of $\widehat{CFA}(Y_1) \otimes \widehat{CFD}(Y_2)$: $u \otimes x$, $v \otimes x$, $t \otimes a$, $t \otimes b$.

$$\begin{tikzcd} u \arrow[d, "m_2(\cdot,\rho_2)"'] \arrow[dr, "1+m_2(\cdot,\rho_{23})"] & \\ x \arrow[r, "m_2(\cdot,\rho_3)"'] & v \end{tikzcd}$$



$$\begin{tikzcd} a \arrow[d, "\rho_3"'] \arrow[dr, "1"] & \\ x \arrow[r, "\rho_2"] & b \end{tikzcd}$$

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$$\begin{aligned} d(t \otimes b) &= t \otimes a + t \otimes \rho_3 x = t \otimes a + t\rho_3 \otimes x = t \otimes a + v \otimes x \\ d(u \otimes x) &= v \otimes x + u \otimes \rho_2 a = v \otimes x + u\rho_2 \otimes a = v \otimes x + t \otimes a \\ d(v \otimes x) &= v \otimes \rho_2 a = v\rho_2 \otimes a = 0 \\ d(t \otimes a) &= 0. \end{aligned}$$

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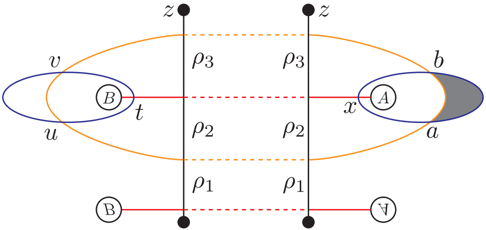

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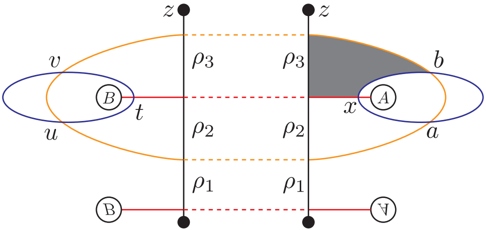

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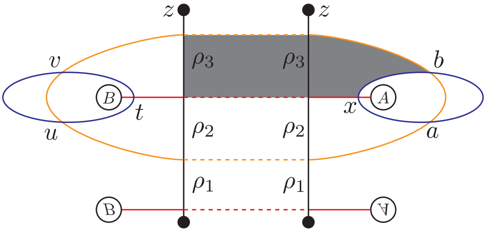

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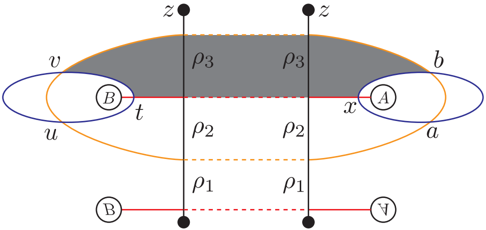

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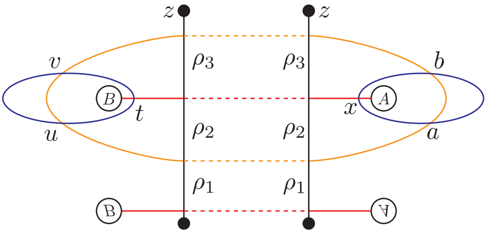
\end{tikzcd}
$$



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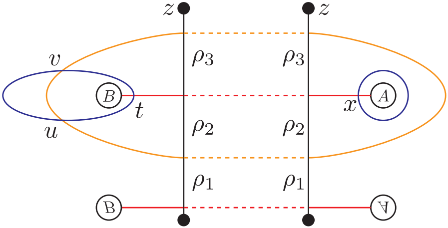
This simplifies to $\mathbb{F}_2\langle t \otimes a + u \otimes x \rangle \oplus \mathbb{F}_2\langle t \otimes b = v \otimes x \rangle$.

$$\begin{tikzcd} u \arrow[d, "m_2(\cdot,\rho_2)"'] \arrow[dr, "1+m_2(\cdot,\rho_{23})"] & \\ x \arrow[r, "m_2(\cdot,\rho_3)"] & v \end{tikzcd}$$



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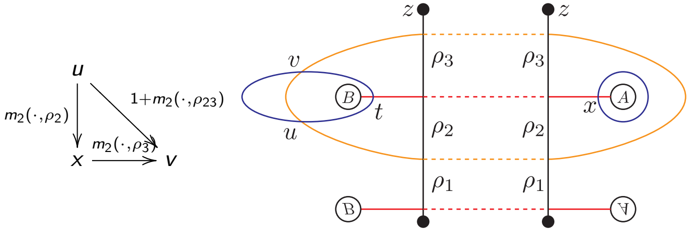

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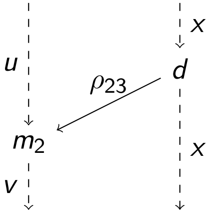
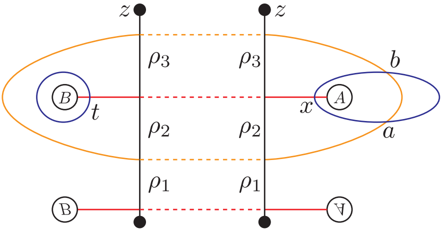
The most interesting part is the interaction:



$$x \xrightarrow{m_3(\cdot,\rho_3,\rho_2)+\dots} x$$



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$$\langle t \otimes a, t \otimes b \mid d(t \otimes a) = t \otimes a + t \otimes b = 0, \quad d(t \otimes a) = 0 \rangle.$$

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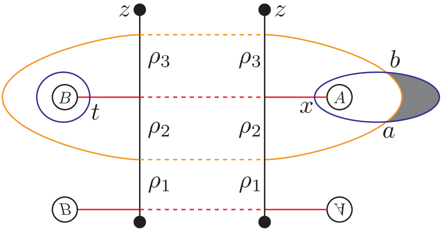

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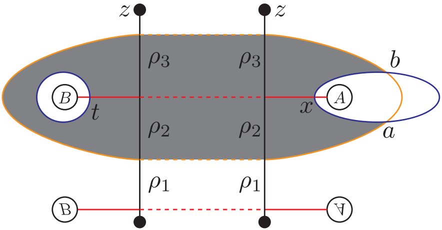

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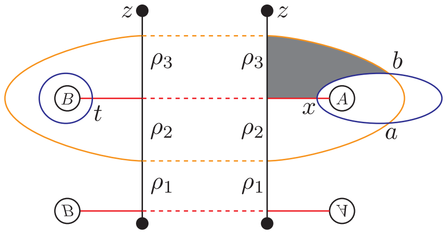

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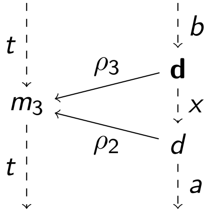

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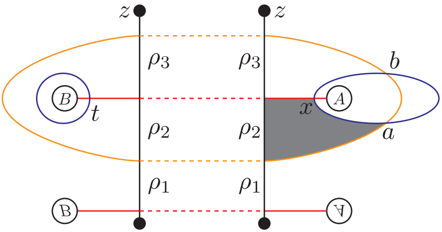

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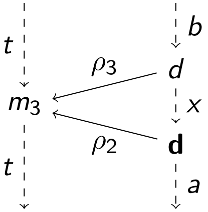

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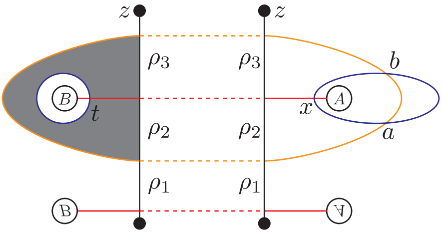

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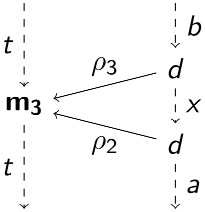

The surgery exact sequence

Theorem

(Ozsváth-Szabó) For K a knot in Y there is an exact sequence

$$\to \widehat{HF}(Y_\infty(K)) \to \widehat{HF}(Y_{-1}(K)) \to \widehat{HF}(Y_0(K)) \to \widehat{HF}(Y_\infty(K)) \to$$

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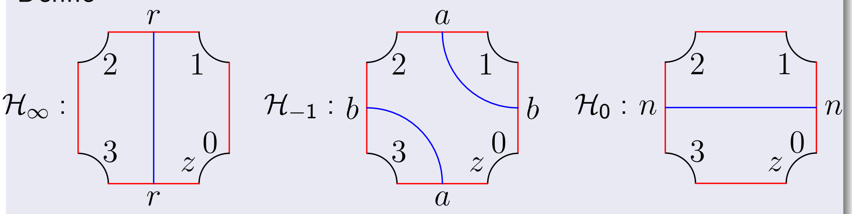
Proof via bordered Floer.

Define



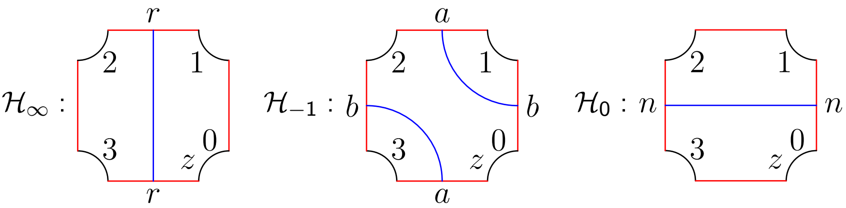
There's a s.e.s.

$$0 \to \widehat{CFD}(\mathcal{H}_\infty) \to \widehat{CFD}(\mathcal{H}_{-1}) \to \widehat{CFD}(\mathcal{H}_0) \to 0.$$

□

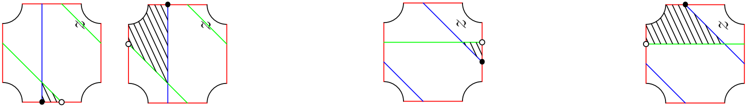
Is it the same sequence?

For



the maps are

$$\varphi(r) = b + \rho_2 a \qquad \psi(a) = n \qquad \psi(b) = \rho_2 n \ .$$



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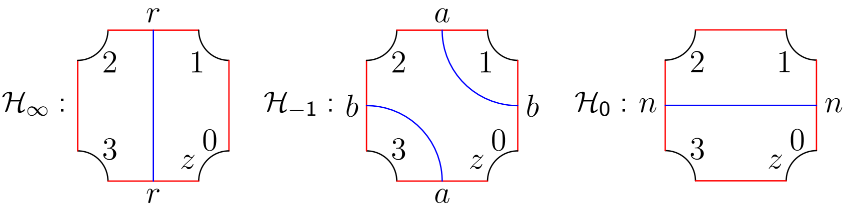

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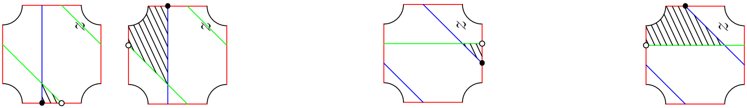

A version of the pairing theorem shows this gives the triangle map on HF.

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- So, this map has a universal definition as a map between $\widehat{CFD}$ of solid tori.
- More generally, the map for attaching handles along a link is given by a concrete map between $\widehat{CFD}$ of handlebodies.