Spaces of diagonal curvature and n-orthogonal coordinate systems

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Plan

1. Spaces of diagonal curvature - elementary theory
2. n-orthogonal coordinate system - new reductions
3. Inverse scattering approach
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1. Spaces of diagonal curvature - elementary theory

\( G^n = \text{n-dimensional Riemannian space admitting a diagonal metric} \)

\[ ds^2 = \sum_{d=1}^{n} H_d^2 d\mathbf{u}_d^2 \]

\( H_d (u_1, \ldots, u_n) \) - Lamé coefficients

\( Q_{ij} \) are rotation coefficients defined as follow

\[ Q_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial u_j} \quad \text{or} \quad \frac{\partial H_j}{\partial u_i} = Q_{ij} H_j; \]

(standard notation \( B_{ij} = Q_{ji} \))

The Ricamann curvature tensor

\[ R_{ij, kl} = 0 \quad i \neq j \neq k \neq l \]

\[ R_{ij, ik} = H_i H_k \left( \frac{\partial Q_{ij}}{\partial u_k} - Q_{ik} Q_{ij} \right) = H_i H_k \left( \frac{\partial Q_{ik}}{\partial u_i} - Q_{ij} Q_{ik} \right) \]

\( i \neq j \neq k \)
$g^i$ is the space of diagonal curvatures if

$R_{ijke} = 0$ for $i \neq j \neq k$

Then

$$\frac{\partial Q_{ij}}{\partial u_k} = Q_{ik} P_{kj}$$

$\psi_i$ - adjoint Lame coefficients

$$\frac{\partial \psi_i}{\partial u_j} = Q_{ji} \psi_j$$

$$\psi_i H = \frac{\partial h}{\partial u_i}$$

$h$ - potential

$$\frac{\partial^2 h}{\partial u_i \partial u_j} = \Gamma_{ike} \frac{\partial h}{\partial u_k} + \Gamma_{jke} \frac{\partial h}{\partial u_k}$$

$\Gamma_{ike} = 0$ for $i \neq j \neq k$

$$\Gamma_{ike} = \frac{1}{H_i} \frac{\partial H_i}{\partial u_k}$$

$$\rho_{ike} = -\frac{H_k}{H_i} \frac{\partial H_i}{\partial u_k}$$
First rank solitonic solution

\[ Q_{ij} = \frac{A_i(u_i) B_j(u_j)}{\Delta} \]

\[ \Delta = \Delta_0 = \sum_{i=1}^{n} \int B_i(\xi_i) A_i(\xi_i) \, d\xi_i \]

\[ \frac{\partial \Delta}{\partial u_i} = - A_i(u_i) B_i(u_i) \]

\[ A_i(u_i), B_i(u_i) \text{ arbitrary functions of one variable} \]

Then

\[ H_i = h_i(u_i) + \frac{A_i(u_i)}{\Delta} K \]

\[ K = K_0 + \sum_{i=1}^{n} \int h_i(\xi_i) B_i(\xi_i) \, d\xi_i \]

\[ \Psi_i = \varphi_i(u_i) + \frac{M_i B_i(u_i)}{\Delta} \]

\[ M = M_0 + \sum_{i=1}^{n} \int \varphi_i(\xi_i) A_i(\xi_i) \, d\xi_i \]

\[ h = h_0 + \sum_{i=1}^{n} \int h_i(\xi_i) \varphi_i(\xi_i) + \frac{K M}{\Delta} \]

\[ K_0, M_0, h_0 \text{ arbitrary constants} \]
By a proper change of variables

\[ U_i \rightarrow U_i(v_i) \]

one may obtain

\[ \mathbf{A}_i = \xi_i \]

Solutions of rank \( k \) are defined as follow

\[ \mathbf{L}_k = \text{ker} \text{ auxillary space} \]

\[ \mathbf{A}_i(v_i) \quad \mathbf{B}_i(v_i) = \text{vectors in } \mathbf{L}_k \]

\[ \mathbf{A}_i = \mathbf{A}_i \rho (v_i), \quad 1 \leq p < k, \quad 1 \leq i \leq n \]

\[ \mathbf{B}_i = \mathbf{B}_i \rho (v_i), \quad 1 \leq p < k, \quad 1 \leq i \leq n \]

\[ \Delta = \text{obstruction in } \mathbf{L}_k \]

\[ \Delta_{pq} = \Delta^0_{pq} + \sum_{i=1}^{n} \int_{\xi_0}^{\xi} \mathbf{B}_i \rho (\xi) \mathbf{A}_i q (\xi) d\xi \]

\[ \Delta^0_{pq} - \text{arbitrary constant matrix} \]

\[ \Delta_{pq} = \Delta^0_{pq} + \sum_{i=1}^{n} \Delta_{pq}^i (v_i) \]

\[ \frac{\partial \Delta_{pq}}{\partial v_i} = - \mathbf{B}_i \rho \mathbf{A}_i q \]
Then
\[ Q_{ij} = A_{ip}(u_i)(D^{-1})_{pq} B_{jq}(u_i) \] - summation over \( p,q \)

solution of rank \( K \). Then
\[ Q_{ij} = \tilde{A}_{ip} \tilde{B}_{pj}(u_i) \]

\[ \tilde{A}_{ip} = A_{ip} \Delta^{-1} \quad \tilde{A}_{ip} = A_{iq}(D^{-1})_{qp} \] - summation

\[ \tilde{A}_{ip} \] is a solution of infinite rank \( K \) - continuous parameter

In solutions of the integral equation
\[ A_{iq} = \int_{-\infty}^{\infty} \tilde{A}_{iq} \Delta_{pq} dq \]

Then
\[ \tilde{A}_{ij} = f_i(u_i) + \tilde{A}_{iq}(p) k_p \]
\[ q_{ij} = g_i(u_i) + \int M_p(D^{-1})_{pq} B_{jq}(u_i) \]
\[ k_p = k_{op} + \sum_{i=1}^{n} u_i \int_{0}^{\infty} f_i(\xi) B_i p(\xi) d\xi \]

\[ m_p = m_{op} + \sum_{i=1}^{n} u_i \int_{0}^{\infty} g_i(\xi) A_i p(\xi) d\xi \]

\[ h = h_0 + \sum_{i=1}^{n} u_i \int_{0}^{\infty} f_i(\xi) g_i(\xi) d\xi + k_p(D^{-1}) \mu q M q \quad \text{summation} \]

In the continuous case we get integration

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_p(D^{-1}) \mu q M q \, dp dq \]

In a general case \( L \) could be

an arbitrary linear space with measure

Notice that \( A_i p \) are combscure

equivalent Lame' coefficients at any \( p \)
2. $\mathbb{R}^n$-orthogonal coordinate system - elementary theory

Let $x^1, \ldots, x^n$ be Cartesian coordinates in $\mathbb{R}^n$.

A curvilinear coordinate system

$u^i = u^i(x^1, \ldots, x^n)$

is orthogonal if

$\sum_{k=1}^{n} \frac{\partial u_i}{\partial x_k} \frac{\partial u^j}{\partial x_k} = h_i^j$.

In new coordinates $R^n$ is the Riemannian space

with the metric $\tilde{h}_{ij} = \frac{1}{h_i^j}$,

$$|ds^2| = h_i^j du^i du^j.$$

This space is flat, it is Riemann.

This is identically equal to zero.

This is a special case of the space of diagonal curvature tensor.

Curvature is a special curvature.

This space is flat, it is Riemann.
Diagonal elements of the Riemann tensor are

\[ R_{ij,ij} = H_{ji} E_{ij} \]  \hspace{1cm} (1.2)

Thus equation \( E_{ij} = 0 \) must be solved together with (1.4)

\[ E_{ij} = \frac{\partial Q_{ij}}{\partial x_j} + \frac{\partial Q_{ji}}{\partial x_i} + \sum_{k \neq i, j} Q_{ik} Q_{jk} \]  \hspace{1cm} (2.3)

To find explicit form one should remember of curvilinear coordinates that the straight lines

\[ x^i (s) = x^i (u^1, u^2, \ldots, u^n) \]

are geodesics in \( \mathbb{R}^n \) for any \( \xi \). Then

\[ \frac{\partial^2 x^i}{\partial u_k \partial u_l} = \Gamma^k_{\ell \kappa} \frac{\partial x^\ell}{\partial u_k} + \Gamma^\ell_{k \kappa} \frac{\partial x^k}{\partial u_l} \]  \hspace{1cm} (2.5)

all \( x^i \) are "potential". Moreover

\[ \frac{\partial^2 x^i}{\partial u_k \partial u_l} = \sum_k \Gamma^k_{\ell \kappa} \frac{\partial x^\ell}{\partial u_k} \]  \hspace{1cm} (2.6)
For a solitonic solution of arbitrary rank

\[ E_{ij} = \tilde{A}_{ij} \tilde{B}_{ip} + \tilde{A}_{ip} \tilde{B}_{ij} + \tilde{A}_{ip} \tilde{A}_{iq} \tilde{U}_{pq} \]

\[ \tilde{U}_{pq} = \sum_{k=1}^{n} \tilde{B}_{kp} \tilde{B}_{kq} \]

If \( A_{ip}, B_{ip} \) are connected by following relation

\[ A_{ip} = \Phi_{pq} B_{iq} \]

where

\[ \Delta_{pq} \Phi_{pq} + \Delta_{pq} \Phi_{qp} = 0 \]

\[ \Delta_{pq} \Phi_{pq} = \Phi_{pq} + \Phi_{qp} = 0 \]

The case \( \Delta_{pq} = \delta_{pq} \) is described by differential reduction in framework of the IST method.
Another example different the IST prescription is presented by the spherical coordinate system in $\mathbb{R}^3$. In this case

$$A_1 = 0 \quad A_2 = \frac{1}{\sin x_2} \quad A_3 = 1$$

$$B_1 = 1 \quad B_2 = \frac{\cos x_2}{\sin x_2} \quad B_3 = 0$$

Now

$$A_2 B_2 = -\frac{\partial}{\partial x_2} \frac{1}{\sin x_2}$$

$$\Delta = \frac{1}{\sin x_2}$$

$$Q_{12} = Q_{13} = Q_{23} = 0$$

$$Q_{21} = 1 \quad Q_{31} = \sin x_2$$

$$x_1 = \rho \quad x_2 = \theta \quad x_3 = \phi$$

$$Q_{12} = \cos x_2$$
To accomplish the solution of construction of the $n$-orthogonal coordinate potentials, one must satisfy an additional equation

$$\frac{\partial^2 h}{\partial \nu_i^2} = \sum \Gamma_{ik} \frac{\partial h}{\partial \nu_k}$$

This equation is equivalent to the condition

$$\frac{\partial^2 \psi_i}{\partial \xi_i^2} = -\sum Q_{ik} \psi_k$$

This condition is satisfied if all $q_i(x_i)$ are constants. An arbitrary choice of $q_i$ coordinates $x_i$ are affine but not orthogonal.

They must be orthogonalized by a proper linear transformation.
Inverse scattering approach (dressing method)

Let $f_i(s)$, $i = 1, \ldots, n$ be functions on $\mathbb{R}^n$.

Let $t + F$ is a linear operator

$$[(t + F)^{\pm}]_i = f_i(s) + \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} F_{ik}(s, s') f_k(s') ds'$$

We consider two factorizations

$$t + F = (1 + k^+)^{-1} (1 + k^-)$$

$$t + F = (1 + M^+)(1 + M^-)^{-1}$$

$k^+(s, s') = 0$ at $s' < s$

$m^+(s, s') = 0$

$k^-(s, s') = 0$ at $s' > s$

$m^-(s, s') = 0$

$$1 + M^+ = (1 + k^+)^{-1}$$
$k^+$ and $m^+$ satisfy the Marchenko integral equations

$$k^+(s, s') + F(s, s') + \int_{s'}^{s} k^+(s, s'') F(s'', s') \, ds'' = 0$$

$$m^-(s, s') + F(s, s') + \int_{s'}^{s} F(s, s'') m^-(s'', s') \, ds'' = 0$$

Moreover

$$k^+(s, s') + m^+(s, s') + \int_{s'}^{s} k^+(s, s'') m^+(s'', s') \, ds'' = 0$$

Let $F$ satisfy differential equations

$$D_i F = 0 \quad D_i F = \frac{\partial F}{\partial u_i} + I_i \frac{\partial F}{\partial s} + \frac{\partial F}{\partial s_i} I_i$$

$I_i, I_j = I_i \delta_{ij}$ are projectors

$I_i = \text{diag} (0, \ldots, 1, \ldots, 0)$

$F_{ij}(s, s', u) = F_{ij}(s-u_i, s'-u_j) \quad F_{ij} = 0$

$N(N-1)$ functions of two variables
Apparently \( [D_i, D_j] = 0 \)

Also
\[
\delta_i k^+ = D_i k^+ + [I_i, Q] k^+ = 0
\]
\[
\delta_i M^- = D_i M^- + M^- [I_i, Q] = 0
\]

Here \( Q = k^+(s, s, u) = M^- (s, s, u) \)

and
\[
\frac{\partial}{\partial u_k} \sigma_i (\omega_i \omega_j) = \sigma_i \omega_i \omega_j
\]

or
\[
\frac{\partial Q_i j}{\partial u_k} = Q_i k Q_k j
\]

Solutions of rank \( K \) are obtained

\[
F_{ij} = \sum_k F_{ik}^+ (s - u_k) \delta_j (s' - u_j')
\]

\[
\frac{\partial Q_i j}{\partial s} + \sum_k \frac{\partial Q_i j}{\partial u_k} = 0
\]
Then
\[ H_i = \Phi_i(s-u_i) + \sum_{k} \int_{s}^{\infty} K^+_k(s,s'u) \Phi_k(s'-u_k) \, ds' \]
\[ \Psi_i = \Psi_i(s-u_i) + \sum_{k} \int_{s}^{\infty} \Phi_k(s'-u_k) M^-_{k_i}(s',s,u) \, ds' \]

Suppose
\[ \frac{\partial F(s,s')}{\partial s'} + \frac{\partial F^{+2}(s',s)}{\partial s} = 0 \]

Then
\[ \left( \frac{\partial K^+(s,s')}{\partial s} \right)_{s'} + \frac{\partial M^-(s,s')}{\partial s} + M^-_{s,s'} \left[ \Phi(s') - \Phi^+(s') \right] = 0 \]

\[ \left( \frac{\partial K^+(s,s')}{\partial s} \right)_{s'} + \frac{\partial K^+(s,s')}{\partial s} \right|_{s'=s} = -\Phi(s) \Phi^+(s) \]

\[ E_{i,j} = 0 \]
\[ E_{ij} = \frac{\partial W_{ij}}{\partial u_i} + \frac{\partial W_{ij}}{\partial u_j} + \sum_{k \neq i, j} Q_{ik} Q_{jk} = 0 \]

The solitonic solution of rank \( K \) appears if

\[ F = \sum_{pq} \frac{\partial F_p (s-u_p)}{\partial s} \Lambda_{pq} \Phi^p (s^1-u) \]

\[ \Lambda_{pq} = -\Lambda_{qp} \quad \text{skew-symmetric constant matrix} \]

To provide satisfaction of relation

\[ \frac{\partial \Psi_i}{\partial x_i} = -\sum_{k \neq i} Q_{ik} \Psi_k \]

one has to put \( q_i (s-u_i) = c_i \) are constants
1. Approach

\[ y = \gamma_{ij} \left( \lambda, \xi \right) \quad i, j = 1, \ldots, n \]

quasia nalytic matrix function on \( \mathbb{C}^+ \)

\[ \frac{\partial y}{\partial \xi} = -\pi i \int y(\zeta) \tilde{f}(\xi, \lambda) d\zeta \]

\[ \frac{\partial y}{\partial \xi} = \pi i \int \tilde{f}(\xi, \zeta) \hat{y}(\zeta) d\zeta \]

\[ \hat{y}(\zeta) = \gamma_{ij}(-\zeta) \quad \hat{f}(\lambda, \zeta) = f(-\xi, -\lambda) \]

\[ f(\xi, \zeta) = e^{i(\xi \Phi + \Phi^0 \xi, \zeta)} e^{-i\lambda \Phi} \]

\[ \Phi = \sum \xi_i \Gamma_i \quad \Gamma_i : \Gamma_j = \Gamma_i \delta_{ij} \quad \text{projector operators} \]

\[ \Gamma_i = \text{diag} \left( 0, \ldots, 1, \ldots, 0 \right) \]

\[ f_{\text{em}} \left( \gamma, \xi \right) = f^0_{\text{em}} e^{i(\lambda \mu \xi + \xi \lambda \mu)} \]

\[ f^0_{\text{em}} - \text{constant matrix} \]
\[ \lambda \rightarrow \lambda + \frac{Q}{\bar{x} \cdot x} + \frac{P}{(\bar{\lambda})^2} \]

\[ \tilde{\lambda} \rightarrow 1 - \frac{Q^{1/2}}{\bar{x} \cdot x} + \frac{P^{1/2}}{(\bar{\lambda})^2} \]

\[ Q = \int f(\bar{x}, \bar{\lambda}, \bar{\mu}) \chi(\bar{x}, \bar{\lambda}, \bar{\mu}) d \bar{x} d \bar{\lambda} d \bar{\mu} \]

\[ P = 1 \tilde{\lambda} \tilde{x} \]

\[ \phi \rightarrow 1 + \frac{B}{\bar{x} \cdot x} \rightarrow \lambda \]

\[ B = -P - P^{1/2} + Q \tilde{Q}^{1/2} \]

The condition

\[ \int \frac{\partial}{\partial \bar{x}} \phi d x d \bar{x} = 0 \rightarrow B = 0 \]
\[ \frac{d p}{d u^p} = \frac{\partial y}{\partial u^p} + i x \frac{\partial y}{\partial \Gamma_p} \]

\[ d_p y \rightarrow i x \Gamma_p + \Omega \Gamma_p \quad \lambda \rightarrow \infty \]

\[ \Gamma_q d_p y \rightarrow \Gamma_q Q \Gamma_p \]

\[ \Gamma_p f = \Gamma_q \partial_{u^q} f - \Gamma_q Q \Gamma_p y = 0 - \text{Lax representation} \]

\[ \Gamma_q \left( \frac{\partial y}{\partial u^p} + i x y \Gamma_p \right) - \Gamma_q Q \Gamma_p f = 0 \]

\[ \Gamma_q \frac{\partial \Omega}{\partial u^p} + \Gamma_q P \Gamma_p - \Gamma_q Q \Omega \Gamma_p \Omega = 0 \]

\[ \frac{\partial \Omega qe}{\partial u^p} = \Omega q p \Omega p e \]

\[ \frac{\partial \Omega q q}{\partial u^p} = \Omega q p \Omega p q \]

\[ P_{p q} = Q q p Q p p - \frac{\partial Q q p}{\partial u^p} \]
\[ \phi = \lambda y k \rightarrow \lambda + \frac{b}{\chi} + \ldots \]

\[ R_{pq} = \frac{\partial \Omega_{pq}}{\partial u_q} + \frac{\partial \Omega_{qp}}{\partial u_p} + \sum_{k \neq p, q} Q_{pk} \Omega_{qk} \]

Let us denote

\[ H(u, \ldots, u_n, \xi, \bar{\xi}) = \mathcal{F}(u, \ldots, u_n, \xi, \bar{\xi}) \quad \xi \neq \bar{\xi} \]

Then

\[ \mathcal{I}_q \frac{\partial H}{\partial u_p} = \mathcal{I}_q \mathcal{I}_p H \]

\[ H_q = \mathcal{I}_q H \]

\[ \frac{\partial H_q}{\partial u_p} = \Omega_{qp} H_p \]

compose a set of combscure equivalent Lame' coefficients, \( \xi \) is a "label"
Then

\[ B = -iπ \int \gamma (\xi, \bar{\xi}) \left( \lambda f_0(\xi, \lambda) - \xi f_1^t(-\lambda, -\bar{\xi}) \right) e^{i \xi p \bar{\xi} + \bar{\xi} p \xi} \, dp d\bar{p} \cdot dp \bar{d} \xi \]

\[ = -iπ \int H(\xi, \bar{\xi}) \left( \lambda f_0(\xi, \lambda) - \xi f_1^t(-\lambda, -\bar{\xi}) \right) H^*(\lambda, \bar{\xi}) \, dp d\bar{p} \cdot dp \bar{d} \xi \]

The condition

\[ \lambda \frac{d}{d\xi} f_0(\xi, \lambda) - \xi f_1^t(-\lambda, -\bar{\xi}) \]

\[ \downarrow \]

\[ B = 0 \]

\[ \downarrow \]

\[ E_{ij} = 0 \quad \text{Flat space!} \]

\[ -iπ \left( \lambda f_0(\xi, \lambda) - \xi f_1^t(-\lambda, -\bar{\xi}) \right) = R(\xi, \lambda) \]

\[ B = \frac{π}{\hbar} \int H(\xi, \bar{\xi}) R(\xi, \lambda) H^*(\lambda, \bar{\xi}) \, dp d\bar{p} \cdot dp \bar{d} \xi \]
5. Shapes of flat normal connection

\[ E_{ij} = \int H_{i p}^{(\xi)} H_{j q}^{(\lambda)} R_{pq} (\xi, \lambda) \, d\xi \, d\lambda \]

\[ R_{pq} (\xi, \lambda) = R_{qp} (\lambda, \xi) \]

One can check

\[ \frac{\partial E_{ijk}}{\partial x_k} = Q_{i \mu} E_{kj} + Q_{j \mu} E_{ki} \]

This is the Biachi identity for diagonal curvature tensor.

Let

\[ R_{pq} (\xi, \lambda) = \sum_{\ell=1}^{N} R_{pq}^{\ell} (\xi, \lambda) \, d\ell \]

\[ H_{pq}^{\ell} = \sum_{\ell=1}^{N} H_{i p}^{\ell} R_{pq}^{\ell} (\xi) \, d\xi \]

\[ E_{ij} = \sum_{\ell=1}^{N} H_{i \ell}^{e} H_{j \ell}^{e} \, d\ell \]
This is the metric of a space of flat normal connection (flat normal bundle).

If \( N \to \infty \) any \( R_{pq}(\mathfrak{g}, \lambda) \) can be obtained. This is a proof of the Frolovian conjecture for spaces of diagonal curvature solvable by the \( \bar{\partial} \) approach.

On an elementary level one can impose the condition

\[ A_{ip} = \mu_{ip} B_{i} q \]

Thus

\[ E_{ij} = \sum A_{ip} A_{iq} R_{pq} \]

\[ R_{pq} = \Delta_{pq} \Psi_{pq} + \Delta_{pqp} \Psi_{q} \]

If \( R_{pq} = \sum R_{pl} R_{ql} \) we obtain the space of flat connection.
6. Can we find new solutions of the Einstein equation? Not, so far - pity!

Let \( n = 4 \), then the Einstein tensor \( C^i_j \) and

\[
C^i_j = \sum_{i \neq j \neq k} C_{ijk}
\]

\[
C^i_{ij} = \frac{E_i}{cH}\frac{E_j}{cH}
\]

Thus any Einstein space of diagonal curvature give a solution of the Einstein equations with some diagonal energy - momentum tensor or dust?

All spherically symmetric solutions of Einstein equations are spaces of diagonal curvature.
remember that
\[ ds^2 = -h_0^2 \, dt^2 + h_1^2 \, dx_1^2 + h_2^2 \, dx_2^2 + h_3^2 \, dx_3^2 \]

In the vacuum case

\[ e_{01} = e_{23} = 1 \quad 1 + \beta + \gamma = 0 \]

\[ e_{02} = e_{13} = \beta \]

\[ e_{03} = e_{12} = \delta \]

No new solutions are found.

Let the metric is synchronous \( h_0 = 1 \)

\[ \Psi + \beta = \frac{dH_2}{d\beta} \frac{1}{H_1^2} \]

Off-diagonal Einstein equations are

\[ \frac{\partial \Psi + \beta}{\partial x^i} = \Psi + \gamma \, \Psi \beta \]

- diagonal curvature condition
\[
\frac{1}{\mathcal{H}_2} \dot{\mathcal{H}}_2 + \frac{1}{\mathcal{H}_3} \dot{\mathcal{H}}_3 = 0 \quad d \neq \beta \neq \gamma
\]

Metric of system of two black holes moving along a straight line must be inside of this metric.

Suppose \( \dot{\mathcal{H}}_2 = 0 \)

\[
F_{23} = F_{13} = \text{are constant in time}
\]

The Lame\' coefficients satisfy to the dynamical system (for evolution of a dust cloud)

\[
\begin{align*}
\ddot{H}_1 + \dot{H}_2 \dot{H}_2 + \dot{H}_1 H_2 &= F_{12} \\
\ddot{H}_1 H_3 + \dot{H}_1 \dot{H}_3 + \dot{H}_1 H_3 &= F_{13} \\
\ddot{H}_2 H_3 + \dot{H}_2 \dot{H}_3 + \dot{H}_2 H_3 &= F_{23}
\end{align*}
\]
This is the Lagrangian system with Lagrangian

\[ L = \dot{H}_1 \dot{H}_2 \dot{H}_3 + H_2 \dot{H}_1 H_3 + H_3 \dot{H}_1 H_2 + F_{12} H_3 + F_{13} H_2 + F_{23} H_1 \]

and Hamiltonian

\[ H = \dot{H}_1 \dot{H}_2 \dot{H}_3 + H_2 \dot{H}_1 H_3 + H_3 \dot{H}_1 H_2 - F_{12} H_3 - F_{13} H_2 - F_{23} H_1 \]

In the vacuum case

\[ H = 0 \]

For the symmetric case

\[ F_{ab} = \pm \frac{1}{k_0^2} \]

one can put

\[ H_i = \alpha_i \dot{h}_i \]

and get the Friedmann equation for the scale factor of Universe

\[ 2 \ddot{a} + \dot{a}^2 = \frac{C}{a^2} \]
If \( H_3 = H_2 \sin X_2 \) one set

\[
Q_{2h} = \frac{\partial H_2}{\partial X_1} h = \sqrt{1 + f(R)} = \text{const.}
\]

\[
Q_{31} = \sin \alpha Q_{21}
\]

\[
Q_{32} = \cos \alpha
\]

\[
H_2 = \frac{1}{\dot{r}}
\]

\[
\dot{R} = f(R) + \frac{F(R)}{R}
\]

Tolman

solution for the collapse of dust cloud.

Remember, that all equation must be solved together with constraints.

Evolution of \( H_3 \) is the "Curvetsuma flow" in a 3-d space of diagonal curvature with given constant rotation coefficients.