FOCK SPACE AS INTEGRAL OVER SPACES ON RANDOM CONFIGURATIONS (PATHS) AND NON-FOCK FACTORIZATIONs. (PARTIALLY WITH M.I.GRAEV)

A. M. VERSHIK (St.Petersburg)

3 мая 2011 г.

CONFERENCE "The VERSALITY OF INTEGRABILITY"
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Celebrating of Igor Krichever’s 60th Birthday
COLUMBIA UNIVERSITY, 4-7 of May 2011, NEW-YORK
1. USUAL MODEL OF THE FOCK SPACE: CONTINUOUS TENSOR PRODUCT OF HILBERT SPACES OR ITO-WIENER SPACE OVER BROWNIAN MOTION.

2. INTEGRAL MODEL OF FOCK SPACE, GENERALIZED LEBESGUE MEASURE IN INFINITE DIMENSIONAL SPACE AND INFINITE DIMENSIONAL CARTAN GROUP.

3. INTEGRAL MODEL OF THE REPRESENTATIONS OF THE CURRENT GROUPS WITH COEFFICIENTS IN THE SEMISIMPLE GROUPS OF RANK ONE: $O(n,1)$ OR $U(n,1)$.

4. APPLICATION TO CURRENT GROUPS ON PARABOLIC SUBGROUPS OF RANK 1 ISOMORPHISM WITH OLD MODEL OF THE REPRESENTATION IN THE FOCK SPACE.

5. NON-FOCK FACTORIZATIONS - BLACK NOISE. 0-DIMENSION (VOTING) MODEL; 1-2 DIMENSIONAL EXAMPLES.
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5. **NON-FOCK FACTORIZATIONS - BLACK NOISE. 0-DIMENSION (VOTING) MODEL; 1-2 DIMENSIONAL EXAMPLES.**
Let $X$ is a manifold with measure $m$; $H$ is an auxiliary Hilbert space.

**Fock space:**

$$H = \bigoplus_{k=0}^{\infty} H \otimes_{\text{sym}} = \text{EXP}H,$$

Suppose that $K$ is another Hilbert space; $X$ is manifold with measure $dx$; if

$$H = \mathcal{L}_2(X;K) \equiv \mathcal{L}_2(X) \otimes K \equiv \int \mathcal{\bigoplus}_{X} K_x dx; K_x \sim K,$$

then we can write (using multiplicative property of EXP):

$$H = \text{EXP}\left\{ \int \mathcal{\bigoplus}_{X} K_x dx \right\} \equiv \int \otimes_{X} K_x dx$$

**BY DEFINITION** this is a continuous tensor product of the Hilbert spaces.
Let $X$ is a manifold with measure $m$; $H$ is an auxiliary Hilbert space.
CANONICAL MODEL: EXP

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BY DEFINITION this is a continuous tensor product of the Hilbert spaces.
$\mathcal{E}^2(\mathbb{X}; K) = \mathbb{L}^2(\mathcal{S}(\mathbb{X}); \nu)$, where right side is $\mathbb{L}^2$ over white noise (gaussian) law $\nu$ (for 1-dimensional case — derivative of the brownian motion). Many-particles decomposition, creation and annihilation operators, product-vectors etc.
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Factorization is the system of the compatible tensor decompositions of operator algebra onto subalgebras, corresponding to the partitions:

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X = \bigcup_{k=1}^{r} X_k; \quad X_k \cap X_{k'} = \emptyset (k \neq k') \quad \Rightarrow \quad \text{END}[\mathcal{H}] = \bigotimes_{k=1}^{r} \text{END}[\mathcal{H}_k]
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*Product vectors or vacuum vectors with respect to given factorization:*
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Product vectors or vacuum vectors with respect to given factorization:
Araki-Woods theorem (1964) and Various Model of Fock Factorization

Necessary and sufficiently condition on factorization to be Fock factorization is there are total set of product vectors. **Corollary**

Let $L^2(\mathcal{S}(\mathcal{M}); \mu)$ where $\mu$ is a law of Levy process on the space $\mathcal{S}(\mathcal{M})$ of Schwartz distributions on the manifold $\mathcal{M}$ with natural factorization is Fock factorization. The vacuum vectors are multiplicative functionals $\xi \mapsto \exp\left\{\int \xi(x) dF(x)\right\}$

One-particle subspace is a space of additive (linear) functional on the process, Probabilistic model of Fock space is the space $L^2$ not over white noise (Ito-Wiener space over $X$) but other Levy measures. Examples: Poisson process; isomorphism of it with Fock space (Gel'fand-Graev-Vershik; Neretin); gamma process (Tsilevich-Yor-V) etc.

product (vacuum)-vectors is exponent of linear functionals. But there are a distinguish example.
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*(Feldman, Tsilevich-V)*

Let $L^2(S(M); \mu)$ where $\mu$ is a law of Levy process on the space $S(M)$ of Schwartz distributions on the manyfold $M$ with natural factorization is Fock factorization. The vacuum vectors are multiplicative functionals $\xi \mapsto \exp\{\int \xi(x)dF(x)\}$
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"Integral model" of bosonic Fock space

Define the cone

\( K(X) = \{ \gamma \} \),

\( \gamma = \{ x_1, c_1 \delta x_1, x_2, c_2 \delta x_2, \ldots \} \),

where

\( c_1 \geq c_2 \geq \cdots \geq 0 \);

\( \sum c_s < \infty \);

\( x_s \in X \).

Define a new Hilbert space:

\( H = \int_{\gamma \in C(X)} \infty \otimes_{s=1} H x_s, c_s \, dL(\gamma) \),

Here we use only countable tensor product and integration over the space of configurations.
"Integral model" of bosonic Fock space

(M.I.Graev-A.V.-2006.)
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Define the cone $\mathcal{K}$ is the cone of all the finite discrete measures on $X$:

$$\mathcal{K}(X) = \{\gamma\}; \gamma = \{x_s, c_s\}_{s=1}^{\infty} = \sum_{s} c_s \delta_{x_s}$$

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Here we use only countable tensor product and integration over the space of configurations.
Definition of the measure $\mathcal{L}$ through Laplace Transform

Laplace transform of a measure:

$$\int_K \exp\{-\langle f, \gamma \rangle\} dL(\gamma) = \exp\{-\theta \int_X \ln f(x) \ dm(x)\}$$

Here $f(x) > 0$ a.e., $\langle f, \gamma \rangle = \int_X f(x) \gamma(x) \ dm(x) \equiv \sum_s f(x_s) c_s \theta > 0$.

For $\theta = 1$ we called the measure $L$ generalized infinite dimensional Lebesgue or stable measure.
Definition of the measure $\mathcal{L}$ through Laplace Transform

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Fundamental property of the Measure $\mathcal{L}_\theta$
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Infinite dimensional Cartan group:

\[ \mathcal{M} = \{ a(.) : \int_X \ln a(x) dx = 0 \quad a(x) \geq 0 \} \]
Fundamental property of the Measure $\mathcal{L}_\theta$

Infinite dimensional Cartan group:

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Action on $\mathcal{K}$:

Theorem

There exist a unique measure (sigma-finite) on the space of Schwartz's distribution $\mathcal{L}_\theta$ such that for any measurable $B_1$.

$$\mathcal{L}_\theta(\mathcal{M} a B_1) = \mathcal{L}_\theta(B_1) \quad \text{(invariance)}$$

$$a(\cdot) \equiv c; \quad \mathcal{L}_\theta(c B) = c \theta \mathcal{L}_\theta(B) \quad \text{(homogeneity)}$$
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$$M_a(\gamma) = M_a(\sum c_s \delta_{x_s}) = \sum c_s a(x_s) \delta_{x_s}$$
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$$1. \mathcal{L}_\theta(M_a B) = \mathcal{L}_\theta(B)$$
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1. $\mathcal{L}_\theta(M_a B) = \mathcal{L}_\theta(B)$ (invariance)

2. $a(\cdot) \equiv c; \mathcal{L}_\theta(c B) = c^\theta \mathcal{L}_\theta(B)$
Fundamental property of the Measure $\mathcal{L}_\theta$

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$$\mathcal{M} = \{a(\cdot) : \int_X \ln a(x) \, dx = 0 \quad a(x) \geq 0\}$$

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Theorem

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(homogeneity)
Group of the coefficients and its representations.

Let $G$ is group of type $R \ast \bowtie G_0 (R^* = \mathbb{R}^+)$.

Let also $\pi_r, r \in R^*$ is a unitary representation of the group $G_0$ in the Hilbert space $K_r$.

Suppose that for different $r$ representations $\pi_r$ are NOT equivalent but are equivariant: there exist ISOMETRY $T_r: K_r \to K_1$ such that:

$$\pi_r(\cdot) = \pi_1(T_r \cdot)$$

When $r \to 0$ representation $\pi_r \to \text{Id}$ - tends to identity representation.
Group of the coefficients and its representations.

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Let $G$ is group of type $\mathbb{R}^* \ltimes G_0$ ($\mathbb{R}^* = \mathbb{R}_+$)
Let also $\pi_r, r \in \mathbb{R}^*$ is a unitary representation of the group $G_0$ in the Hilbert space $K_r$.
Suppose that for different $r$ representations $\pi_r$ are NOT equivalent but are equivariant: there exist ISOMETRY $T_r : K_r \to K_1$ such that:

$$\pi_r(.) = \pi_1(T_r.)$$
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When $r \rightarrow 0$ representation $\pi_r \rightarrow Id$ - tends to identity representation.
Representation of the current group

Define the current group of the bounded measurable functions on the manifold $X$ with values in $G$:

$$G^X = \{ x \mapsto g(x) \in G \}$$

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Construction of the representation of the current group

STEP 1. Choose trajectory (configuration) $\gamma = \sum c_s \delta x_s$; $\sum s c_s < \infty c_1 \geq \cdots \geq 0$

For each $\gamma$ define a Hilbert space which is countable tensor product $\bigotimes_s K c_s$ in which we have presentation of $\times x_s G_0$.

Consider the numbers $c_s > 0$ and define the COUNTABLE tensor product $\bigotimes_s K c_s$.

Let current $g(x) \in G_0, x \in X$. Now we correspond to the configuration $\gamma$ and current $g(x)$ the operator in the space $\bigotimes_s K c_s$: $\Pi_\gamma (g(x)) = \bigotimes s \pi c_s g(x_s)$.
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Construction of the representation of the current group

\[ R \ast X \ni r(\vec{x}) \text{ change the charges} \]

\[ \sum c_s \delta \vec{x}_s \mapsto \sum r(\vec{x}_s) c_s \delta \vec{x}_s \]

So tensor products \( \bigotimes c_s \) over configuration \( \gamma \) goes to tensor product \( \bigotimes r(\vec{x}_s) c_s \) over configuration of \( \gamma_r(\cdot) \). Consequently we change operators \( \Pi_{\gamma}(g(\cdot)) \) of representations \( \bigotimes c_s \) in the space \( \bigotimes c_s \) onto operators \( \Pi_{\gamma_r}(g(\cdot)) \).

This is possible because of equivariance of representations of \( \pi_r \).
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Construction of the representation of the current group

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So tensor products $\bigotimes_s K_{c_s}$ over configuration $\gamma$ goes to tensor product $\bigotimes_s K_{r(x_s)c_s}$ over configuration of $\gamma^r(.)(.)$, consequently we change operators $\Pi_\gamma(g(.)$) of representations $\bigotimes_s \pi_{c_s}$ in the space $\bigotimes_s K_{c_s}$ onto operators $\Pi_{\gamma^r}(g(.))$. 
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Integration

STEP 3. Now we can integrate over all configurations $\gamma$ over generalize Lebesgue measure $L$:

$$H = \int_{\gamma \in \mathbb{K}(X)} \bigotimes_{s=1}^{\infty} K_{x_s}, c_s \, dL(\gamma),$$

IMPORTANT. Measure $L$ is invariant with respect to multiplication on $r(x)$ iff

$$\int \ln r(x) = 0.$$

We obtain the representation $\Pi$ of the group $G_X$.

Theorem. The representation $\Pi$ is irreducible.

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**Theorem**

*The representation $\Pi$ is irreducible.*

**Доказательство.**

The group $\mathbb{R}^*_X$ has ergodic action on $\mathcal{K}(X)$.
Example: $O(n, 1)$, $U(n, 1)$ and parabolic its subgroups.
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Maximal parabolic subgroup $P \subset O(n, 1)$ is isomorphic to the group of triples

$$(r, u, c), \quad \text{where} \quad r \in \mathbb{R}^*, \quad u \in O(n - 1), \quad c \in \mathbb{R}^{n-1}$$
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$$(r, u, c), \quad \text{where} \quad r \in \mathbb{R}^*, \quad u \in O(n-1), \quad c \in \mathbb{R}^{n-1}$$

with multiplication

$$(r_1, u_1 c_1) (r_2, u_2, c_2) = (r_1 r_2, u_1 u_2, c_1 + r c_2 u).$$

So this group $P$ is semisimple product

$$P = \mathbb{R}^* \rtimes P_0, \quad \text{where} \quad P_0 = O(n-1) \rtimes \mathbb{R}^{n-1},$$

and elements $r \in \mathbb{R}^*$ acts on $P_0$ as the automorphisms

$$(u, c) \mapsto (u, c)^r = (u, r c).$$
Extension on $O(n, 1), U(n, 1)$
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Theorem
Consider $K_r = L^2(B_r)$, where $B_r$ is Euclidean of the radius $r$ with usual representation $\pi_r$ of the motion group $P_0 = M_{n-1}$. Then the construction above gives the unitary representation of the current group $P^X$ of the bounded measurable functions on the manifold $X$ with values in the parabolic group $P$, and this representation naturally extends onto current group $O(n, 1)^X$. 
Theorem
Consider \( K_r = L^2(B_r) \), where \( B_r \) is Euclidean of the radius \( r \) with usual representation \( \pi_r \) of the motion group \( P_0 = M_{n-1} \). Then the construction above gives the unitary representation of the current group \( P^X \) of the bounded measurable functions on the manifold \( X \) with values in the parabolic group \( P \), and this representation naturally extends onto current group \( O(n,1)^X \). The case of the group \( U(n,1) \) is similar.
Difficult Task: To construct factorization without product (=vacuum) vectors (or with rare set of its).

("Extremely Non-Additive conjunction")

Examples for \(d = 0, d = 1\).

A. Versik & B. Tsirel'son (1998),
\(d = 2\): S. Smirnov-O. Schramm (2009). (Percolation model)
\(d = 0\) \text{--- is the simplest (zero-dimensional) example: A CANTOR set}
\(X = \prod_{1}^{\infty} \{0, 1, 2\}\) \text{--- the set of the infinite path in the triadic tree } T_3 \text{ with one root.}
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Election by majority
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The Hilbert space is $\mathcal{H} = L^2(X)$ with a measure $\mu$.

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The imbedding

$$\psi_n : H_n = \mathbb{C}^{2^{3n-1}} \rightarrow H_{n+1} = \mathbb{C}^{2^n}, n = 1, \ldots$$

is tensor product of one imbedding ($n = 1$)
Election by majority

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$$\psi : \mathbb{C}^2 \rightarrow \mathbb{C}^8.$$
The map $\psi$ is linear map corresponded to map $2 \rightarrow 8$ ($8 = 2^3$):
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Basis in $\mathbb{C}^2$ — $e_1, e_2$, in $\mathbb{C}^8$ — $e^1 \ldots e^8$. 

Theorem

The factorization of the $L^2$ by cylindric sets over space of the pathes of triadic tree has no product vectors besides constant and consequently defines a Non-Fock factorization.
The map $\psi$ is linear map corresponded to map $2 \rightarrow 8$ ($8 = 2^3$): Basis in $\mathbb{C}^2 = e_1, e_2$, in $\mathbb{C}^8 = e^1 \ldots e^8$. Each vertex of cube corresponds to vector of basis:

$e_1 \sim (0), e_2 \sim (1); e^1 \sim (0, 0, 0), e^2 \sim (1, 0, 0), e^3 \sim (0, 1, 0), e^4 \sim (0, 0, 1), e^5 \sim (1, 1, 0), e^6 \sim (1, 0, 1), e^7 \sim (0, 1, 1), e^8 \sim (1, 1, 1)$

$f(e^1) = f(e^2) = f(e^3) = f(e^4) = e_1 \quad f(e^5) = f(e^6) = f(e^7) = f(e^8) = e_2$

$\psi(e_1) = e^1 + \ldots + e^4; \quad \psi(e_2) = e^5 + \ldots + e^8$
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**Theorem**

*The factorization of the $L^2$ by cylindric sets over space of the paths of triadic tree has no product vectors besides constant and consequently defines a Non-Fock factorization.*

Discussion.
References


