The Versatility of Integrability
Celebrating Igor Krichever's 60th Birthday

Quantum Integrability
and
Gauge Theory

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This is a work on experimental theoretical physics

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Hep-th [arXiv:1103.3919]
Earlier work
G.Moore, NN, S.Shatashvili,
arXiv:hep-th/9712241;
A.Gerasimov, S.Shatashvili.
NN, S.Shatashvili,
NN, E.Witten
arXiv:1002.0888

Earlier work on instanton calculus
A.Losev, NN, S.Shatashvili,
NN, arXiv:hep-th/0206161;
The papers of
N.Dorey, T.Hollowood, V.Khoze, M.Mattis,

Earlier work on separated variables and D-branes
A.Gorsky, NN, V.Roubtsov,
arXiv:hep-th/9901089;
In the past few years a connection between the following two seemingly unrelated subjects was found.
The supersymmetric gauge theories with as little as 4 supersymmetries on the one hand
Quantum integrable systems and soluble by Bethe Ansatz on the other
The **supersymmetric vacua** of the (finite volume) **gauge theory** are **the stationary states** of a quantum integrable system.
The « twisted chiral ring » operators

$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \ldots, \mathcal{O}_n$

map to the quantum Hamiltonians

$H_1, H_2, H_3, \ldots, H_n$
**Eigenvalues**

The vacuum expectation values of the twisted chiral ring operators identify with the energy and other eigenvalues on the integrable side.

\[ E_k(\lambda) = \langle \lambda | O_k | \lambda \rangle \]

Identify with the energy and other eigenvalues on the integrable side.

\[ H_k \Psi_\lambda = E_k(\lambda) \Psi_\lambda \]
The main ingredient of the correspondence:

The effective twisted superpotential of the gauge theory

= The Yang-Yang function of the quantum integrable system
The effective twisted superpotential of the gauge theory

\[ A = a + \vartheta^+ \psi_+ + \bar{\vartheta}^- \bar{\psi}_- + \vartheta^+ \bar{\vartheta}^- (F_A + iD) \]

\[ \mathcal{L}^{\text{eff}} = g_{ij} d_{ai} \wedge * d_{aj} + g^{ij} \left( \text{Re} \left( \frac{\partial \tilde{W}}{\partial a_i} \right) \text{Re} \left( \frac{\partial \tilde{W}}{\partial a_j} \right) + F_i \wedge * F_j \right) + i \text{Im} \left( \frac{\partial \tilde{W}}{\partial a_i} \right) F_i \]
The effective twisted superpotential leads to the vacuum equations

\[ \exp \frac{\partial \tilde{W}(a)}{\partial a_i} = 1 \]
The effective twisted superpotential

\[ \tilde{W}^{\text{eff}}(a_1, \ldots, a_N; \varepsilon; \tau, m) \]

Is a multi-valued function on the Coulomb branch of the theory, depends on the parameters of the theory.
The Yang-Yang function of the quantum integrable system

The YY function was introduced by C.N.Yang and C.P.Yang in 1969 for the non-linear Schroedinger problem.
The miracle of Bethe ansatz: The spectrum of the quantum system is described by a classical equation

\[ \exp \left( \frac{\partial \tilde{W}(a)}{\partial a_i} \right) = 1 \]
EXAMPLE: Many-body system

Calogero-Moser-Sutherland system

\[ H_{eCM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + g^2 \sum_{i<j} U(x_i - x_j; q) \]

\[ p_k = -i\hbar \frac{\partial}{\partial x_k} \]
The elliptic Calogero-Moser system

\( \forall \) identical particles on

a circle of radius \( \beta \)

subject to the two-body interaction

elliptic potential

\[
U(x; q) = U(-x; q) = \sum_{n \in \mathbb{Z}} \frac{1}{\sinh^2(x + 2\pi n \beta)}
\]
Quantum many-body systems

One is interested in the $\beta$-periodic symmetric, $L^2$-normalizable wavefunctions

$$\Psi(x_1, \ldots, x_N)$$
It is clear that one should get an infinite discrete energy spectrum.
Many-body system vs gauge theory

The infinite discrete spectrum of the integrable many-body system = The vacua of the $N=2$ $d=2$ theory
The gauge theory

The $N=2\ d=2$ theory, obtained by subjecting the $N=2\ d=4$ theory to an $\Omega$-background in $\mathbb{R}^2$

$\Phi \rightarrow \Phi - \varepsilon D_\varphi$
The four dimensional gauge theory on $\Sigma \times \mathbb{R}^2$, viewed $SO(2)$ equivariantly, can be formally treated as an infinite dimensional version of a two dimensional gauge theory.
The two dimensional theory

Has an effective twisted Superpotential!
The effective twisted superpotential

\[ \tilde{W}^{\text{eff}}(a_1, \ldots, a_N; \varepsilon; \tau, m) \]

Can be computed from the \( \text{N=2 d=4} \)
Instanton partition function

\[ Z(a, \varepsilon_1, \varepsilon_2; m, \tau) \]
The effective twisted superpotential

\[ Z(a, \varepsilon_1, \varepsilon_2; m, \tau) \sim e^{\frac{1}{\varepsilon_2}} \tilde{W}^{\text{eff}}(a_1, \ldots, a_N; \varepsilon_1; \tau, m) + \ldots \]

as \( \varepsilon_2 \to 0 \)

\[ \Phi \to \Phi - \varepsilon_1 \, D_{\varphi_1} - \varepsilon_2 \, D_{\varphi_2} \]
The effective twisted superpotential has one-loop perturbative and all-order instanton corrections

\[ \widetilde{W}_{\text{eff}}(a; \tau) = \widetilde{W}^{\text{pert}}(a) + \sum_{n=1}^{\infty} q^n \widetilde{W}_{n-\text{inst}}(a) \]
In particular, for the $N=2^*$ theory
(adjoint hypermultiplet with mass m)
$N=2^*$ theory

$$\exp \frac{\partial \tilde{W}^{\text{pert}}(a)}{\partial a_i} =$$

$$e^{\frac{\pi i \tau a_i}{\varepsilon}} \prod_{j \neq i} S(a_i - a_j)$$

$$S(x) = \frac{\Gamma \left( \frac{-m+x}{\varepsilon} \right)}{\Gamma \left( \frac{-m-x}{\varepsilon} \right)} \frac{\Gamma \left( 1 - \frac{x}{\varepsilon} \right)}{\Gamma \left( 1 + \frac{x}{\varepsilon} \right)}$$
Bethe equations
Factorized S-matrix

\[ S(x) = \frac{\Gamma \left( \frac{-m+x}{\varepsilon} \right)}{\Gamma \left( \frac{-m-x}{\varepsilon} \right)} \frac{\Gamma \left( 1 - \frac{x}{\varepsilon} \right)}{\Gamma \left( 1 + \frac{x}{\varepsilon} \right)} \]

This is the two-body scattering
In hyperbolic Calogero-Sutherland
Bethe equations
Factorized S-matrix

\[ S(x) = \frac{\Gamma \left( \frac{-m+x}{\varepsilon} \right)}{\Gamma \left( \frac{-m-x}{\varepsilon} \right)} \frac{\Gamma \left( 1 - \frac{x}{\varepsilon} \right)}{\Gamma \left( 1 + \frac{x}{\varepsilon} \right)} \]

\[ U_0(x) = \frac{1}{\sinh^2(x)} \]

Two-body potential
Bethe equations
Factorized S-matrix

\[ S(x) = \frac{\Gamma\left(\frac{-m+x}{\varepsilon}\right) \Gamma\left(1 - \frac{x}{\varepsilon}\right)}{\Gamma\left(\frac{-m-x}{\varepsilon}\right) \Gamma\left(1 + \frac{x}{\varepsilon}\right)} \]

Harish-Chandra, Gindikin-Karpelevich, Olshanetsky-Perelomov, Heckmann,
final result: Opdam
The full superpotential of $N=2^*$ theory leads to the vacuum equations

Momentum phase shift

$$e^{\frac{\pi i \tau a_i}{\epsilon}} \prod_{j \neq i} S(a_i - a_j) \times \left[ 1 + q \sum \prod \text{rational}(a_i, a_l, a_k, m(m + \epsilon), \epsilon) + \ldots \right]$$

Two-body scattering

The finite size corrections

$$q = \exp \left( - N\beta \right)$$
Dictionary
Dictionary

Elliptic CM system $\leftrightarrow N=2\ast$ theory
Dictionary

classical
Elliptic
CM
system

\[\leftrightarrow\]

4d \( N=2^* \)
theory
Dictionary

quantum Elliptic CM system

4d $N=2^*$ Theory in 2d $\Omega$-background
Dictionary

The (complexified) system
Size $\beta$

$\iff$

The gauge coupling $\tau$
Dictionary

The Planck constant

The Equivariant parameter \( \varepsilon \)
The correspondence extends to other integrable systems: Toda, relativistic Systems, perhaps all 1+1 iQFTs
Two ways of getting two dimensional theory starting with a higher dimensional one

1) Kaluza-Klein reduction, e.g. compactification on a torus with twisted boundary conditions…

2) Boundary theory, localization on a cosmic string…. 
Two ways of getting two dimensional theory starting with a higher dimensional one

1) Kaluza-Klein reduction: gives the spin chains, e.g. XYZ
2) Boundary theory, localization on a cosmic string: gives the many-body systems, e.g. CM, more generally, a Hitchin system
Plan

Now let us follow
The quantization procedure
more closely,
Starting on the gauge theory side
In the mid-nineties of the 20th century it was understood, that

The geometry of the moduli space of vacua of \( N=2 \) supersymmetric gauge theory is identified with that of a base of an algebraic integrable system

*Donagi, Witten, Gorsky, Krichever, Marshakov, Mironov, Morozov*
Liouville tori

$M_c^{2n}, \omega_c^2$

The Base
\[ a_i = \oint_{A_i} d^{-1} \omega_c \]
\[ a^i_D = \oint_{B_i} d^{-1} \omega_c \]
\[ a^i_D = \frac{\partial \mathcal{F}}{\partial a_i} \]
The moduli space of vacua of d=4 N=2 theory

Special coordinates on the moduli space.
The Coulomb branch of the moduli space of vacua of the d=4 N=2 supersymmetric gauge theory is the base of a complex (algebraic) integrable system.
The Coulomb branch of the same theory, compactified on a circle down to three dimensions is the phase space of the same integrable system.
This moduli space is a hyperkahler manifold, and it can arise both as a Coulomb branch of one susy gauge theory and as a Higgs branch of another susy theory. This is the 3d mirror symmetry.
The moduli space of vacua of $d=3$ $N=4$ theory
In particular, one can start with a six dimensional (0,2) ADE superconformal field theory, and compactify it on $\Sigma \times S^1$ with the genus $g$ Riemann surface $\Sigma$. 
The resulting effective susy gauge theory in three dimensions will have 8 supercharges (with the appropriate twist along $\Sigma$).
The resulting effective susy gauge theory in three dimensions will have the Hitchin moduli space as the moduli space of vacua. The gauge group in Hitchin’s equations will be the group of the same A,D,E type as in the definition of the (0,2) theory.
The Hitchin moduli space is the Higgs branch of the 5d gauge theory compactified on $\Sigma$.
The mirror theory, for which the Hitchin moduli space is the Coulomb branch, is conjectured by Gaiotto, in the $A_1$ case, to be the $SU(2)^{3g-3}$ gauge theory in 4d, compactified on a circle, with some matter hypermultiplets in the tri-fundamental and/or adjoint representations.
One can allow the Riemann surface with \( n \) punctures, with some local parameters associated with the punctures. The gauge group is then \( SU(2)^{3g-3+n} \) with matter hypermultiplets in the fundamental, bi-fundamental, tri-fundamental representations, and, sometimes, in the adjoint.
For example, the SU(2) with $N_f=4$ corresponds to the Riemann surface of genus zero with 4 punctures. The local data at the punctures determines the masses.
For example, the N=2* SU(2) theory corresponds to the Riemann surface of genus one with 1 punctures. The local data at the puncture determines the mass of the adjoint.
From now on we shall be discussing these « generalized quiver theories »

• The integrable system corresponding to the moduli space of vacua of the 4d theory is the SU(2) Hitchin system on The punctured Riemann surface $\Sigma$
Hitchin system

Gauge theory on a Riemann surface

The gauge field $A_\mu$ and the twisted Higgsfield $\Phi_\mu$ in the adjoint representation are required to obey:
Hitchin equations

\[ \bar{\partial}_\bar{z} \Phi_z + [A_{\bar{z}}, \Phi_z] = 0 \]

\[ \partial_z \Phi_{\bar{z}} + [A_z, \Phi_{\bar{z}}] = 0 \]

\[ F_{zz\bar{z}} + [\Phi_z, \Phi_{\bar{z}}] = 0 \]
Hitchin system

Modulo gauge transformations:

$$(A_\mu, \Phi_\mu) \rightarrow (g^{-1}A_\mu g + g^{-1}\partial_\mu g, g^{-1}\Phi_\mu g)$$

We get the moduli space $\mathcal{M}_H$
Hyperkahler structure of $\mathcal{M}_H$

- Three complex structures: $I, J, K$

- Three Kahler forms: $\omega_I$, $\omega_J$, $\omega_K$

- Three holomorphic symplectic forms: $\Omega_I$, $\Omega_J$, $\Omega_K$
Hyperkahler structure of $\mathcal{M}_H$

Three Kahler forms: $\omega_I, \omega_J, \omega_K$

Three holomorphic symplectic forms:

$$\Omega_I = \omega_J + i \omega_K,$$
$$\Omega_J = \omega_K + i \omega_I,$$
$$\Omega_K = \omega_I + i \omega_J.$$
Hyperkahler structure of $\mathcal{M}_H$

\[
\omega_I = \int_{\Sigma} \text{Tr} \left( \delta A \wedge \delta A + \delta \Phi \wedge \delta \Phi \right)
\]

\[
\omega_J = \int_{\Sigma} \text{Tr} \left( \delta A \wedge * \delta \Phi \right)
\]

\[
\omega_K = \int_{\Sigma} \text{Tr} \left( \delta A \wedge \delta \Phi \right)
\]

\[
\Omega_I = \int_{\Sigma} \text{Tr} \left( \delta \Phi_z \wedge \delta A_{\bar{z}} \right) \, d^2z
\]

\[
\Omega_J = \int_{\Sigma} \text{Tr} \left( \delta A \wedge \delta A \right)
\]

$\mathcal{A} = A + i \Phi$
Hyperkähler structure

The linear combinations, parametrized by the points on a twistor two-sphere $S^2$

\[ a \text{l} + b \text{j} + c \text{k}, \text{ where} \]

\[ a^2 + b^2 + c^2 = 1 \]
The integrable structure

In the complex structure I,
the holomorphic functions are: for each Beltrami differential $\mu^{(i)}$, $i=3g-3+n$

$$H_i = \int \sum \mu^{(i)} \text{Tr} \Phi_z^2$$
The integrable structure

These functions Poisson-commute w.r.t. $\Omega_i$

\[
\{ H_i, H_j \} = 0
\]
The integrable structure

The generalization to other groups is known, e.g. for $G=SU(N)$

\[ H_{p,i} = \int_{\Sigma} \nu_{[p]}^{(i)} \text{Tr} \Phi^p \]

\[ \nu_{[p]}^{(i)} \in H^1(\Sigma, K^{(1-p)}_{\Sigma}) \]

\[ i = 1, \ldots, (2p - 1)(g - 1), \quad p = 2, \ldots, N \]
The integrable structure

The action-angle variables:

Fix the level of the integrals of motion,

ie fix the values of all $H_i$'s

Equivalently:

fix the (spectral) curve $C$ inside $T^*\Sigma$

$\text{Det}(\lambda - \Phi_z) = 0$

Its Jacobian is the Liouville torus, and

The periods of $\lambda dz$ give the special coordinates $a_i, a_D^i$
The quantum integrable structure

The naïve quantization, using that in the complex structure $I$

\[ \mathcal{M}_H \text{ is almost } = \mathcal{T}^*\mathcal{M} \]

Where $\mathcal{M}=\text{Bun}_G$

$\Phi_z$ becomes the derivative

$H_i$ become the differential operators.

More precisely, one gets the space of twisted (by $K^{1/2}_M$) differential operators on $\mathcal{M}=\text{Bun}_G$
Thinking about the \( \Omega \)-deformation of the four dimensional gauge theory, leads to the conclusion that the quantum Hitchin system is governed by a Yang-Yang function. The effective twisted superpotential

\[
\widetilde{W}(a_1, \ldots, a_{3g-3+n}; m_1, \ldots, m_n, \tau_1, \ldots, \tau_{3g-3+n}; \varepsilon)
\]
Here comes the experimental fact

The effective twisted superpotential, the $YY$ function of the quantum Hitchin system:

In fact has a classical mechanical meaning!
\( \mathcal{M}_H \) as the moduli space of \( G_C \) flat connections

In the complex structure \( J \) the holomorphic variables are:

\[
\mathcal{A}_\mu = A_\mu + i \Phi_\mu
\]

which obey (modulo complexified gauge transformations):

\[
\mathcal{F} = d\mathcal{A} + [\mathcal{A},\mathcal{A}] = 0
\]
$\mathcal{M}_H$ as the moduli space of $G_C$ flat connections

In this complex structure $\mathcal{M}_H$ is defined without a reference to the complex structure of $\Sigma$

$$\mathcal{M}_H = \text{Hom} \left( \pi_1(\Sigma), G_C \right)/G_C$$
\( \mathcal{M}_H \) as the moduli space of \( G^C \) flat connections

However \( \mathcal{M}_H \)

Contains interesting complex Lagrangian submanifolds which do depend on the complex structure of \( \Sigma \)

\( L_{\Sigma} = \) the variety of \( \text{G-opers} \)

*Beilinson, Drinfeld*

*Drinfeld, Sokolov*
\[ \mathcal{M}_H \] as the moduli space of \( G_C \) flat connections

\[ \mathcal{L}_\Sigma = \text{the variety of } G\text{-opers} \]

\[ A_z = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad A_{\bar{z}} = \begin{pmatrix} -\frac{1}{2} \partial \mu & \mu \\ \mu T - \frac{1}{2} \partial^2 \mu & \frac{1}{2} \partial \mu \end{pmatrix} \]

The Beltrami differential \( \mu \) is fixed, the projective structure \( T \) is arbitrary, provided

Beilinson, Drinfeld
Drinfeld, Sokolov
\[ M_H \] as the moduli space of \( G_C \) flat connections

\[ \mathcal{L}_\Sigma = \text{the variety of } G\text{-opers} \]

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The Beltrami differential \( \mu \) is fixed, the projective structure \( T \) is arbitrary, provided it is compatible with the complex structure defined by

\[ \bar{\partial} - \mu \partial \]
\( \mathcal{M}_H \) as the moduli space of \( G_C \) flat connections

\[ L_\Sigma = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad A_z = \begin{pmatrix} -\frac{1}{2} \partial \mu \\ \mu T - \frac{1}{2} \partial^2 \mu \\ \frac{1}{2} \partial \mu \end{pmatrix} \]

i.e.

\[
(\bar{\partial} - \mu \partial - 2 \partial \mu) T = -\frac{1}{2} \partial^3 \mu
\]
For example, on a two-sphere with $n$ punctures, these conditions translate to the following definition of the space of opers with regular singularities: we are studying the space of differential operators of second order, of the form

$$-\partial^2 + T$$

$$T = \sum_{i=1}^{n} \frac{\Delta_i}{(z-z_i)^2} + \frac{\varepsilon_i}{z-z_i}$$
Opers on a sphere

Where $\Delta_i$ are fixed, 

$$\Delta_i = \nu_i(\nu_i - 1)$$

while the accessory parameters $\varepsilon_i$ obey

$$\sum_{i=1}^{n} \varepsilon_i = 0$$

$$\sum_{i=1}^{n} z_i \varepsilon_i + \Delta_i = 0$$

$$\sum_{i=1}^{n} z_i^2 \varepsilon_i + z_i \Delta_i = 0$$
Opers on a sphere

All in all we get a (n-3)-dimensional subvariety in the 2(n-3) dimensional moduli space of flat connections on the n-punctured sphere with fixed conjugacy classes of the monodromies around the punctures:

\[ m_i = \text{Tr} \left( g_i \right) \]

\[ m_i = 2 \cos(2\pi \nu_i) \]
The main conjecture

The YY function is the generating function of the variety of opers
The variety of opers is Lagrangian with respect to $\Omega_j$. 
We shall now construct a system of Darboux coordinates on $M_H$.
\[ \alpha_i, \beta_i \]

\[ \Omega_J = \sum_{i=1}^{3g-3+n} d\alpha_i \wedge d\beta_i \]
So that \( L_\Sigma \) = the variety of G-opers, is described by the equations

\[
\beta_i = \frac{1}{\varepsilon} \frac{\partial \tilde{W}}{\partial \alpha_i} \\
\alpha_i = \varepsilon \alpha_i
\]
The moduli space is going to be covered by a multitude of Darboux coordinate charts, one per every pair-of-pants decomposition (and some additional discrete choice)
Equivalently, a coordinate chart $U_{\Gamma}$ per maximal degeneration $\Gamma$ of the complex structure on $\Sigma$. 
The maximal complex structure degenerations = The weakly coupled gauge theory descriptions of Gaiotto theories, e.g. for the previous example

\[ G = SU(2)^9 \]
The $\alpha_i$ coordinates are nothing but the logarithms of the eigenvalues of the monodromies around the blue cycles:

$$\text{Tr} \, P \, \exp \int_{C_i} A = 2 \cosh(\alpha_i)$$
The $\alpha_i$ coordinates are nothing but the logarithms of the eigenvalues of the monodromies around the blue cycles:

\[
\text{Tr} P \exp \int_{C_i} A = 2 \cosh(\alpha_i)
\]

cf. Drukker, Gomis, Okuda, Teschner; Verlinde; Verlinde
The $\beta_i$ coordinates are defined from the local data involving the cycle $C_i$ and its four neighboring cycles (or one, if the blue cycle belongs to a genus one component):
The local data involving the cycle $C_i$ and its four neighboring cycles:
The local picture:
four holes, two interesting holonomies

\[ \text{Hol}_{C^v} \sim g_3 g_2 \]

\[ \text{Hol}_C \sim g_2 g_1 \]
Complexified hyperbolic geometry:
The coordinates $\alpha_i, \beta_i$ can be thus explicitly expressed in terms of the traces of the monodromies:

$$B = \text{Tr} \left( \text{Hol}_C^v \sim g_3 g_2 \right)$$

$$m_i = 2 \cos(2\pi \nu_i) = \text{Tr} g_i$$

$$A = 2 \cosh(\alpha) = \text{Tr} \left( \text{Hol}_C \sim g_2 g_1 \right)$$

$$\frac{B(A^2 - 4) + 2(m_2 m_3 + m_1 m_4) - A(m_1 m_3 + m_2 m_4)}{\sqrt{c_{12}(A)c_{34}(A)}} = 2 \cosh(\beta)$$

$$c_{ij}(A) = A^2 + m_i^2 + m_j^2 - Am_i m_j - 4$$
The construction of the hyperbolic polygon generalizes to the case of $n$ punctures:
The construction of the hyperbolic polygon generalizes to the case of $n$ punctures:
For $g_i$ obeying some reality conditions, e.g. SU(2), SL(2,R), SU(1,1), SO(1,2), or, $R^3$

We get the real polygons in $S^3$, $H^3$, $R^{2,1}$, $E^3$ our coordinates reduce to the ones studied by

Klyachko, Kapovich, Millson, Kirwan, Foth,

**NB:** The Loop quantum gravity community (Baez, Charles, Rovelli, Roberts, Freidel, Krasnov, Livin, .... ) uses different coordinates
Our polygons sit in the group manifold

An interesting problem:
Relate our coordinates to the coordinates
Introduced by Fock and Goncharov,
Based on triangulations of the Riemann surface with punctures.
Our polygons sit in the group manifold

An interesting problem:
Relate our coordinates to the coordinates
Introduced by Fock and Goncharov,
Based on triangulations of the Riemann surface with punctures.

The FG coordinates are the basis of the Gaiotto-Moore-Neitzke work on the hyperkahler metric on $\mathcal{M}_H$. 
The local data involving the cycle $C_i$ on the genus one component

$$\text{tr} \left( g_1 g_2 g_1^{-1} g_2^{-1} \right) = m$$

$$A = \text{tr} (g_1) = 2 \cosh(\alpha),$$

$$B = \text{tr} (g_2) = \left( e^{\frac{\beta}{2}} + e^{-\frac{\beta}{2}} \right) \sqrt{\frac{A^2 - m - 2}{A^2 - 4}}$$
The canonical transformations
(the patching of the coordinates)
The s-t channel flop: the generating function is the hyperbolic volume
The \( s-t \) channel flop: the generating function is the hyperbolic volume

\[
\frac{\partial V^v(\alpha, \alpha', \nu_1, \nu_2, \nu_3, \nu_4)}{\partial \alpha} = \beta
\]

\[
\frac{\partial V^v(\alpha, \alpha', \nu_1, \nu_2, \nu_3, \nu_4)}{\partial \alpha'} = \beta'
\]

\( V^v \) is the volume of the dual tetrahedron
The s-u flop = composition of the 1-2 exchange (a braid group action) and the flop.

The 1-2 braiding acts as:

$(\alpha, \beta)$ goes to $(\alpha, \beta \pm \alpha + \pi i)$
The theory

Why did the twisted superpotential turn into a generating function?
Why did the variety of opers showed up?
What is the meaning of Bethe equations for quantum Hitchin in terms of this classical symplectic geometry?
Why did these hyperbolic coordinates (which generalize the Fenchel-Nielsen coordinates on Teichmuller space and Goldman coordinates on the moduli of SU(2) flat connections) become the special coordinates in the two dimensional N=2 gauge theory?
What is the relevance of the geometry of hyperbolic polygons for the M5 brane theory?
For the three dimensional gravity?
For the loop quantum gravity?
Why did the twisted superpotential turn into a generating function, and why did the variety of opers showed up?

This can be understood by viewing the 4d gauge theory as a 2d theory with an infinite number of fields in two different ways \((NN+EW)\).
The theory

What is the meaning of Bethe equations for quantum Hitchin in terms of this classical symplectic geometry?

They seem to describe an intersection of the brane of opers with another \((A,B,A)\) brane, a more conventional Lagrangian brane. The key seems to be in the Sklyanin’s separation of variables \((\text{NSR})\)

The full YY function is the difference of the generating function of the variety of opers and the generating function of the topological Lagrangian brane (independent of the complex structure of \(\Sigma\))
The theory

Why did these hyperbolic coordinates (which generalize the *Fenchel-Nielsen* coordinates on Teichmuller space and *Goldman* coordinates on the moduli of SU(2) flat connections) become the special coordinates in the two dimensional N=2 gauge theory?

The key seems to be in the relation to the Liouville theory and the SL(2,\(\mathbb{C}\)) Chern-Simons theory. A concrete prediction of our formalism is the quasiclassical limit of the Liouville conformal blocks:

\[
\Psi_{\Gamma}(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n}; b) \sim \exp \frac{1}{b^2} \tilde{W}_{\Gamma}(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n})
\]
The quasiclassical limit of the Liouville conformal blocks (motivated by the AGT conjecture, but it is independent of the validity of the AGT):

\[ \Psi_{\Gamma}(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n}; b) \sim \exp \frac{1}{b^2} \tilde{W}_{\Gamma}(\alpha_1, \ldots, \alpha_{3g-3+n}; m_1, \ldots, m_n; q_1, \ldots, q_{3g-3+n}) \]
In the genus zero case it should imply the Polyakov’s conjecture (proven for Fuchsian m’s by Takhtajan and Zograf); can be compared with the results of Zamolodchikov, Zamolodchikov; Dorn-Otto.
The theory vs experiment

The conjecture in gauge theory has been tested to a few orders in instanton expansion for simplest theories (g=0,1), and at the perturbative level of gauge theory for all theories. What is lacking is a good understanding of the theories with tri-fundamental hypermultiplets (in progress, NN+V.Pestun)
The prediction of the theory

The conjecture implies that the Twisted superpotential transforms under the S-duality in the following way:

\[
\tilde{W}(\tilde{\alpha}; \mu_1 \pm \mu_4, \mu_2 \pm \mu_3; 1 - q) = \\
\text{Crit}_\alpha \left( \tilde{W}(\alpha; \mu_1 \pm \mu_2, \mu_3 \pm \mu_4; q) + V^\vee(\alpha, \tilde{\alpha}; \mu_1, \mu_2, \mu_3, \mu_4) \right)
\]

a generalization of the four dimensional electric-magnetic transformation of the prepotential
FOR THE N-BODY ELLIPTIC CALOGERO-MOSER SYSTEM:

The ingredients of the previous story: the YY function, the 4d gauge theory calculation, the variety of opers are all known. What is missing is the analogue of our $\alpha, \beta$ Darboux coordinates.

To be continued....
For the rest of the puzzles there remains much to be said, hopefully in the near future.

Thank you!