Elliptic Calogero-Moser systems for crystallographic complex reflection groups.

P. Etingof
MIT

joint work with G. Felder, X. Ma, and A. Veselov,
math/1003.4689
(J. of algebra, C. De Concini 60-th birthday volume)
1. Complex reflection groups.
Let \( W \) be a finite group acting faithfully on a f.d. complex vector space \( \mathcal{V} \).
One says that \( W \) is a complex reflection group if it is generated by elements \( s \) such that \( rk(1-s) = 1 \).

Chevalley's theorem: \( W \) is a complex reflection group iff \( \mathbb{C}[\mathcal{V}]^W \) is a polynomial algebra (or, equivalently, \( S_\mathcal{V}^W \) is a polynomial algebra).

Irreducible complex reflection groups were classified by Shepard and Todd in 1954.

Examples: 1) Weyl groups (and more generally, Coxeter groups) 2) \( S_n \times (\mathbb{Z}/2) \mathbb{Z}^n, \mathcal{V} = \mathbb{C}^n \).
2. Crystallographic complex reflection groups.

Definition. A complex reflection group \( W \) is crystallographic if it preserves a lattice \( L \) in \( \mathfrak{h} \) (of rank \( 2 \dim \mathfrak{h} = \dim \mathfrak{r} \)).

Examples. 1) Weyl groups (due to Popov).

2) \( S_n \times (\mathbb{Z}/r\mathbb{Z})^n \) with \( r = 1, 2, 3, 4, 6 \) (we use triangular lattice for \( r = 3, 6 \) and square lattice for \( r = 4 \)).

3. Elliptic Calogero-Moser systems.

Let \( W \) be a crystallographic complex reflection group preserving a lattice \( L \subset \mathfrak{h} \). Then we can consider the complex torus \( X = \mathfrak{h}/L \), which carries an action of \( W \).
If \( W \) acts irreducibly on \( G \), this torus is automatically an abelian variety (so let us assume this).

Let \( s \in W \) be a complex reflection. Let \( X^s \) be the fixed set of \( s \) on \( X \). This is a union of hypertori in \( X \) which are called reflection hypertori.

**Example.**
1) \( W = \mathbb{Z}/2\mathbb{Z} \) acting on \( G = \mathbb{C} \).
   - The 4 pts of order 2 are the reflection hypertori in \( E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}) \) (elliptic curve).

2) \( W = \mathbb{Z}/3\mathbb{Z} \), \( G = \mathbb{C} \)
   - 3 points are reflection hypertori.

3) \( W = \mathbb{Z}/4\mathbb{Z} \), \( G = \mathbb{C} \)
   - 3 points are reflection hypertori.

4) \( W = \mathbb{Z}/6\mathbb{Z} \), \( G = \mathbb{C} \)
   - 3 points are reflection hypertori.
If $T \subset X^5$ is a reflection hypertorus, then we can define the subgroup $W \subset W$ of elements which preserve all points of $T$. This subgroup is isomorphic to $\mathbb{Z}/m_T\mathbb{Z}$ for some $m_T \geq 2$.

Let $\mathfrak{A}$ be the set of pairs $(T,j)$, where $T$ is a reflection hypertorus, and $1 \leq j \leq m_T-1$ is an integer.

There is a natural action of $W$ on $\mathfrak{A}$. Let $C = C[\mathfrak{A}]^W$.

**Example.**
1) $W = \mathbb{Z}_2$. The orders of the 4 points are 2, so $\dim C = 4$.
2) $W = \mathbb{Z}_3$. The orders of the 3 points are 3 so $\dim C = 6$ (E6).
3) $W = \mathbb{Z}_4$. The orders of the 3 points are 2, 4, 4, so $\dim C = 7$ (E7).
4) $W = \mathbb{Z}_6$. The orders of the 3 points are 2, 3, 6, so $\dim C = 8$ (E8).
Main result. (with Felder, Ma, Veselov) 2010

There exists a family of classical and quantum integrable systems, parametrized by $c \in \mathbb{C}$, defined on the abelian variety $X$, whose symbols are generators of the polynomial algebra $(SG)^w$, $P_1, \ldots, P_n$ (where $n = \dim G$). These systems have meromorphic coefficients on $X$ with poles on reflection hypertori. The quantum systems have regular singularities, and their monodromy gives rise to representations of generalized Cherednik (= double affine Hecke) algebras.

Examples. D $W$ is a Weyl group, $B$ its standard representation.
In this case the Hamiltonian of the quantum system is
\[ H = \Delta - \sum_{\alpha > 0} C_{\alpha} (C_{\alpha} + 1)(\alpha, \alpha) \mathcal{P}(\alpha, x, 2) \]
on \( X = \frac{\mathfrak{h}}{\mathcal{L}} = E \otimes \mathcal{P}^v \) dual weight lattice
This is the elliptic Calogero-Moser system.

2) \( W = S_n \times \mathbb{Z}_2^n, \mathfrak{h} = \mathbb{C}^n, L = \mathbb{Z}^n \oplus \mathfrak{h} \)
\( X = E_{\mathfrak{c}}^n \). The Hamiltonian of the quantum system depends on 5 parameters and is given by
\[ H = \Delta - \sum_{i \neq j} k(k+1)(\alpha_i x_i - x_j, 2) + \alpha_i (x_i + x_j, 2) \]
- \[ \sum_{e=1}^{n} \sum_{j=1}^{n} C_e (C_e + 1) \mathcal{P}(x_j - \frac{x_i}{3} e, 2) \]
where \( \frac{x_i}{3} e \) are 10 points of order 2.
This is the Inozemtsev system, which is a deformation of example (where we had 2 parameters).
In the $n=1$ case there are 4 parameters, and we get the Darboux operator, which is a 4-parameter generalization of the Lamé operator:

$$L = D^2 - m(m+1)\Delta(x,t)$$

3) New example. Let $W = S^n \times \mathbb{Z}_r^n$ where $r = 3, 4, 6$. Then $X = E^n_\mathbb{Z}$, where $E_\mathbb{Z}$ is an elliptic curve with special symmetry. Then the basic invariants of $W$ are $\Sigma p_i^r$, $\Sigma p_i^2$, ..., $\Sigma p_i^{mr}$. So the lowest Hamiltonian has symbol $\Sigma p_i^r$. Let us give the explicit expression for $r = 3$.

In this case we have seven parameters $R, a_0, b_0, a_1, b_1, a_2, b_2$, and the Hamiltonian looks like
\[ H = \sum_{i=1}^{n} \alpha_i^3 + \sum_{i=1}^{n} \left( a_0 \varphi(x_i) + a_1 \varphi(x_i - \eta_1) \right) + a_2 \varphi(x_i - \eta_2) \] 

\[ -3k(k+1) \sum_{i<j} \sum_{p=0}^{2} \varphi(x_i - \varepsilon^p x_j)(\varepsilon_i + \varepsilon^p \varepsilon_j) \] 

\[ + \sum_{i=1}^{n} \left( b_0 \varphi'(x_i) + b_1 \varphi'(x_i - \eta_1) + b_2 \varphi'(x_i - \eta_2) \right) \]

where \( \varepsilon = \varepsilon = e^{2 \pi i/3}, \varphi(x) = \varphi(x, \tau), \eta_1 = i \sqrt{3}/3, \eta_2 = -i \sqrt{3}/3. \)


We will use the method of Buchstaber-Felder-Veselov, closely related to the method of Cherednik. This method is based on the notion of Elliptic Dunkl Operators.
4a. **Elliptic Dunkl Operators**

Fix a generic line bundle $L \in \mathcal{P}(\mathcal{M})$, such that $L \not\equiv \mathcal{O} \ (\text{globally}).$

For every $(T, j) \in \mathcal{A}$, there is a global meromorphic section $f_{T, j}$ of the vector bundle

$$f_{T, j} (s_{T}^{-1} \otimes \mathcal{O}_{T})$$

which has a simple pole at $T$ with residue 1 and no other singularities.

(residue is well defined since we can interpret sections of this bundle as 1-forms, as $b_{T}^{*} \leq b_{*}^{*}$).

Let $C$ be a $W$-invariant function on $\mathcal{A}$. Let $v \in \mathcal{B}$.

**Definition.** The elliptic Dunkl operator

$$D_{v} \overset{\Delta}{=} D_{v} + \sum_{(T, j) \in \mathcal{A}} C(T, j)f_{T, j}(v, \mathcal{O}) s_{T}^{-1}$$
Example. Let \( W \) be a Weyl group. Then the elliptic Dunkl operators are the operators defined by BFV:

\[
D_{\nu, c}^{\lambda} = D_{\nu} + \sum_{\alpha \neq 0} \left( C_{\nu}(\alpha, \nu) \delta_{\nu}(\alpha, \nu) \sigma_{\alpha}(\nu) \right),
\]

where

\[
\sigma_{\mu}(z) = \frac{\theta(z - \mu) \theta'(-\mu)}{\theta(z) \theta(-\mu)}, \quad \forall \mu \in \mathbb{R}^*.
\]

The elliptic Dunkl operators act on sections of \( \mathcal{X} \).

Theorem. (E.-Ma)

1) \( [D_{\nu, c}^{\lambda}, D_{\nu, c}^{\mu}] = 0 \)

2) \( g \circ D_{\nu, c}^{\lambda} \circ g^{-1} = D_{\nu, c}^{g \lambda} \)

This gives rise to the idea of proof of the main result which goes back to BFV. The idea is to construct the integrals of the elliptic crystallographic \( MN \) system by applying the symmetric polynomials \( P_i \) to the elliptic Dunkl operators.
Reminder on classical Dunkl and Calogero-Moser operators.

$s \subset \mathcal{W}$ set of reflections $c: S \rightarrow C$ is a $\mathcal{W}$-invariant function.

$$D^0 \nu, c = P \nu + \sum_{s \in S} \frac{2c(s)\alpha_s(\nu)}{(1 - \beta_s)\alpha_s} s,$$

where $\beta_s$ is the nontrivial eigenvalue of $s$ on $\mathfrak{g}^*$, and $\alpha_s$ are the roots corresponding to $s$.

$$P_i^c(p, \lambda) = m(P_i(D^0, c)),$$

where $m(\sum \varrho g \varphi) = \sum \varrho \varphi$.

**Remark.** In fact don't need $m$ (as $P_i^c(p, \lambda)$ is already a function). But this is not obvious and not true in the quantum case.
However, there is a problem with this approach: the elliptic Dunkl operators depend on the line bundle $\mathcal{L}$ and are not $W$-invariant: under the action of $W$, $\mathcal{L}$ goes to $g \mathcal{L}$. They have a chance of becoming $W$-invariant only when $\mathcal{L}$ tends to the trivial bundle, but in this limit they develop a singularity (recall that $\mathcal{L}$ had to be "generic"). So the idea has to be modified, so that we are able to "subtract" these singularities and successfully pass to the limit $\mathcal{L} \to$ trivial bundle. To this end, it turns out that classical rational Calogero-Moser system comes handy.

5. To explain this method, let us parametrize line bundles $\mathcal{L}$ on $X$ by points $\lambda \in \mathcal{B}$, identifying...
identifying $\mathfrak{g}$ with its Hermitian dual $\mathfrak{g}^*$ by means of a $W$-invariant positive Hermitian inner product $\mathfrak{B}$ on $\mathfrak{g}$ (if $W$ is irreducible, this inner product is unique up to scaling).

We will write this parametrization as $\lambda \mapsto \lambda_\lambda$, and denote $D^\lambda_\lambda$ by $D_\lambda^\lambda, c$. Also let $P^c_\lambda(p, \lambda)$ be the rational classical Calogero–Moser Hamiltonians attached to $(W, \mathfrak{g})$ and parameter $c$ (which is a function on the set of reflections in $W$ invariant under conjugation).

**Theorem 1.** For an appropriate linear function $c = c(C, \lambda)$, the operators $L^c_\lambda = P^c_\lambda(c)(D_\lambda^\lambda, c, \lambda)$ are holomorphic near $\lambda = 0$. 
2) The operators $L_i = L_i^C$ of a $W$-invariant pairwise commuting elements of $CW \times D(X_{reg})$, where $X_{reg}$ is the complement of reflection hypertori in $X$.

3) The restrictions $L_i^C$ of $L_i^C$ to $W$-invariant meromorphic functions on $X$ are commuting differential operators on $X_{reg}$, whose symbols are the polynomials $P_i$.

The operators $L_i^C$ are the Hamiltonians of the crystallographic elliptic Calogero-Moser system.

The classical version of this system is constructed by the classical version of this construction.
6. A geometric construction of the same system (in the style of Beilinson & Drinfeld).

Let $X$ be a complex alg. variety, and $W$ a finite group acting faithfully on $X$.
For $g \in W$, consider the fixed set $x_g$. If $y \in x_g$ is a connected component, it is called a reflection hypersurface.

Given $y$, we have a cyclic group $Wy \leq W$ of elements fixing $y$ pointwise, whose order we will call $n_y$. Let $\mathfrak{g}$ be the set of pairs $(y,j)$, where $y$ is a reflection hypersurface, and $j=1,\ldots,n_y-1$. Let $c$ be a $W$-invariant function on $\mathfrak{g}$, and let $y$ be a twisting for differential operators on $X$ (as in Beilinson-Bernstein).
In 2004, to this data I attached a quasicoherent sheaf of algebras on $X/\mathfrak{w}$, denoted $H_{c,\Psi,\mathfrak{w},x}$.

On affine open sets $U$, the algebras of sections $H_{c,\Psi,\mathfrak{w},x}(U)$ are generated by $\mathcal{O}_x$, $\mathfrak{w}$, and "generalized Dunkl operators". Also, defining $e = \frac{1}{||c||} \sum_{g \in \mathfrak{w}} e$, we define a sheaf $eH_{c,\Psi,\mathfrak{w},x}e$. These sheaves are called the sheaf of Cherednik algebras and the sheaf of spherical Cherednik algebras.

**Remark.**
1) $H_{0,\Psi,\mathfrak{w},x} = \mathcal{O}W \times D_{\Psi}(X)$;
2) $H_{c,\Psi,\mathfrak{w},x} = \mathcal{O}W \times D_{\Psi}(X)$ after localization on the reflection hypersurfaces.
3) If $X = \mathfrak{g}$ is a linear representation of $\mathfrak{w}$ then $H_{c, \psi, \mathfrak{w}, x}(x) = H_{c}(\mathfrak{w}, \mathfrak{g})$ is the usual rational Cherednik algebra.

4) If $U$ is the formal neighborhood of a point $\bar{x} \in X/\mathfrak{w}$ then $H_{c, \psi, \mathfrak{w}, x}(U)$ is Morita equivalent to the rational Cherednik algebra of the stabilizer $\mathfrak{w}_x$ of a preimage $x \in X$ of $\bar{x}$, acting on $T_x X$.

Now let us come back to the situation when $X$ is an abelian variety and $\mathfrak{w}$ an irreducible crystallographic reflection group. In this case twistings $\psi$ lie in $H^2(X)^W = \mathbb{C}^W$, where $\mathfrak{w}$ is the Kähler form.
Theorem. There is a linear function \( \ell : C[[\mathfrak{g}]]^n \to C \cdot \omega \) such that:
\[
\Gamma(eH_\psi, c, w, x e) = C \quad \text{unless} \quad \psi = \ell(c) \quad \text{("critical level condition")}
\]
In this case, \( \Gamma(eH_\psi, c, w, x e) \) is a polynomial algebra in \( n \) generators. Evaluating this algebra in \( C W_\chi D_\psi (X_{reg}) \) after localization, we get the elliptic crystallographic \( \mathcal{CM} \) system.

This is analogous to the Beilinson-Drinfeld construction of the quantum Hitchin system in the geometric Langlands theory:

\[
\text{quantum Hitchin} = \Gamma(D_{-\chi}^n(\text{Bun}_E^x))
\]
7. Applications to finite-gap operators (with E. Rains).

From the work of Airault, McKea, and Moser it's known that elliptic finite-gap potentials are parametrized by critical points of elliptic Calogero-Moser Hamiltonians. More generally, Krichever showed that elliptic CM flow provides elliptic solutions of the KP hierarchy. This application is toward extension of these results to operators with symmetries.

Let $E$ be a special elliptic curve with $Z_r$ symmetry, $r = 2, 3, 4, 6$. Let $L$ be a differential operator on $E$ of the form

$$\partial^r + a_{r-2}(z) \partial^{r-2} + \ldots + a_0(z)$$
which is \( \mathbb{Z}_r \)-invariant, has poles at the fixed points \( \eta_i \) of \( \mathbb{Z}_r \) and also some points \( \varepsilon_1, \ldots, \varepsilon_n \) (taken from distinct \( \mathbb{Z}_r \)-orbits) together with their images. Let \( r_j \) be the order of stabilizer of \( \eta_j \) and let \( L \) behave near \( \eta_j \) as \( L_{r_j} \), where \( L \) is a homogeneous operator with any integer indices. Also let us fix the indices at the poles \( \varepsilon_1, \ldots, \varepsilon_n \) to be the "smallest interesting case": 

\[-1, 1, \ldots, e-2, e.\]

**Conjecture.** Such operators correspond to critical points of the elliptic crystallographic CM lowest Hamiltonian.
For $r=2$, this is classical (Inozemtsev system). For $r=3$, this is checked in my paper with Rains. Open for $r=4,6$.

Generalization to non-integer indices at $\eta$: require trivial monodromy only around $2j$, $j=1,\ldots,N$ for $I\psi = \lambda\psi$.

Generalization to higher Hamiltonians:
Consider operators of order $nr$ with special conditions at poles.

Also have rational degenerations of these conjectures which make sense for any $r \in \mathbb{N}$.
Happy birthday, Igor Moiseevich!