ALGEBRAIC CURVES AND NON-LINEAR DIFFERENCE EQUATIONS

I.M. Kričhever

In [1] we have given an account of a scheme for the integration of certain non-linear differential equations by methods of algebraic geometry. After a slight modification, the main ideas and results of the scheme can be carried over to difference equations.

1. Let

\[ L_1^{ij} = \sum_{\alpha = -n_1}^{n_1} u_{\alpha} (\xi) \delta_i j, \quad \xi = j - \alpha, \quad L_2^{ij} = \sum_{\beta = -m_1}^{m_1} v_\beta (\xi) \delta_i j, \quad j - \beta \]

be difference operators whose coefficients are \((l \times l)\)-matrices. We stipulate that their highest and lowest coefficients are non-singular diagonal matrices with distinct diagonal elements.

We consider equations in the coefficients of these operators that are equivalent to the equality \([L_1, L_2] = 0\).

The operator \(L_2\) induces on the solution space of the equation \(L_1 y = Ey\) a finite-dimensional linear operator \(L_2(E)\). Its characteristic polynomial \(Q(w, E)\) defines a complex curve \(\mathfrak{R}\), and the projection \((w, E) = P \to E\) defines a meromorphic function on it.

**THEOREM 1.** For any pair of commuting difference operators we can find a polynomial in two variables such that \(Q(L_2, L_1) = 0\).

If all the eigenvalues of \(L_2(E)\) are distinct, as in the case of pairwise coprime numbers \(n_2, m_2\) and \(n_1, m_1\), then to each point \((w, E)\) of \(\mathfrak{R}\) there corresponds an eigenvector of \(L_2(E)\) that is unique up to a proportionality factor.

**THEOREM 2.** If \((n_2, m_2) = 1\) and \((n_1, m_1) = 1\), then \(E(P)\) has \(l\) poles \((P_1, \ldots, P_l)\) of order \(n_2\) and \(l\) poles \((P_1', \ldots, P_l')\) of order \(n_1\). The coordinates \(\psi_j(i, P)\) of the eigenvector-functions of \(L_1\) and \(L_2\) belong to the space associated with the divisor \(\Delta = D + (i - 1)D_{\infty} + P_i - P_i'\), where \(D\) is an effective divisor whose degree \(g\) is equal to the genus of the curve for almost all solutions of the original equations, and \(D_{\infty} = (P_1' + \ldots + P_l') - (P_1 + \ldots + P_l)\).

We consider the inverse problem of recovering the operators from a curve with distinguished points \(P_j\) and a divisor \(D\) of degree \(g\).

Since \(\deg \Delta = g\), by the Riemann–Roch theorem the \(\psi_j(i, P)\) are uniquely determined by the conditions of Theorem 2 up to a normalization. Having fixed one, we have the following theorem.

**THEOREM 3.** For any function \(E(P)\) with poles on \(\mathfrak{R}\) only at the points \(P_j\), there exists a unique operator \(L\) such that \(L \psi_j(i, P) = E(P) \psi_j(i, P)\).

2. In this section we construct exact solutions for certain non-linear differential-difference equations.

Suppose that we are given a set of polynomials \(Q_j^k\) and \(R_j^k\) (k).

**THEOREM 4.** For every effective divisor \(D\) on a curve \(\mathfrak{R}\) of genus \(g\) (\(\deg D = g\)) with fixed local coordinates \(k_j^1(P)\) in neighbourhoods of the \(P_j\), one and (apart from a proportionality factor) only one there exists function \(\psi_j(k, y, t, P)\) that is meromorphic outside \(P_j^2\), and for which \(D\) is the divisor of the poles. In a neighbourhood of \(P_j^2\) the function

\[ \psi_j (t, \ y, \ t, \ P) \exp \left( Q_j^k (k_j^1 (P)) y + R_j^k (k_j^1 (P)) t \right) \]

has a pole (zero) of order \(i\) if \(j = j_1\), and of order \(i - 1\) if \(j \neq j_1\).

By defining the normalization of \(\psi_j(k, y, t, P)\) arbitrarily we obtain the vector-valued function \(\psi(i, y, t, P)\).

**THEOREM 5.** There exist unique difference operators whose coefficients depend on \(y\) and \(t\), such that

\[ \left( L_1 - \frac{\partial}{\partial y} \right) \psi (s, \ y, \ t, \ P) = 0 \quad \text{and} \quad \left( L_2 - \frac{\partial}{\partial t} \right) \psi (s, \ y, \ t, \ P) = 0. \]

**COROLLARY.** These operators satisfy the equation
Communications of the Moscow Mathematical Society

\[ [L_1, L_2] = \frac{\partial L_2}{\partial y} - \frac{\partial L_1}{\partial t}. \]

3. **Example.** We consider the equations of a Toda chain:

\[ \dot{v}_n = c_{n+1} - c_n, \quad c_n = c_n (v_n - v_{n-1}). \]

By Theorem 4, there is a unique function \( \psi(n, t, P) \) with poles at the points \( d_1, \ldots, d_g \) of \( \mathcal{R} \) defined by \( w^2 = \prod_{i=1}^g (E - E_i) \), and with the following asymptotic expansion at the inverse images of \( E = (P^\pm) \):

\[ \psi^\pm (n, t, E) = \eta n^{\pm \frac{1}{2}} E^{\pm \frac{1}{2}} (1 + \xi^\pm (n, t) E^{-1} + \ldots) \exp \left( \mp \frac{1}{2} tE \right). \]

By Theorem 5, the operators

\[ L^{nm} = i \sqrt{c_n \delta_{m,n+1} + v_n \delta_{n,m-1} - c_n \delta_{m,n}} \]
\[ A^{nm} = i \sqrt{c_n \delta_{m,n+1} + w_n \delta_{n,m-1} + \frac{i}{2} c_n \delta_{m,n}} \]

satisfy the equations \( L \psi = \psi \) and \( A \psi = \delta \psi / \delta t \). Here \( \sqrt{c_n} = \lambda_n^{1/2}/\lambda_n \), \( v_n = \xi^1 (n+1, t) - \xi^1 (n, t) \), and \( w_n = v_n/2 + \lambda_n/\lambda_n \).

The equations (1) are equivalent to the system

\[ \frac{c_n}{c_n} = (v_n - v_{n-1}) - (w_n - w_{n-1}) = \frac{1}{2} (v_n - v_{n-1}) - \frac{1}{2} \frac{c_n}{c_n}, \]

which is the same as the equations of a Toda chain.

We must remark that this representation of equations is different from the commutation representation, used in earlier work (for a bibliography, see [2]).

By expressing \( \psi(n, t, P) \) in terms of Riemann's theta-function as in the formula of Its [3] and also § 3 of [1], we obtain the following formulae in which we have used the notation of [1]:

\[ \log c_n = \frac{d}{dn} \log \theta (\omega^* + W) \theta ((n-1) U + tV + W + \omega^*) \theta (\omega^* - W) \theta ((n-1) U + tV + W - \omega^*) + \text{const}, \]
\[ v_n = \frac{d}{dn} \frac{d}{dt} \log \theta (nU + tV + \omega^* + W) \theta (tV + \omega^* - W) + \text{const}, \]

where the vectors \( \omega^*, V, \) and \( U \) and the constants depend only on the curve \( \mathcal{R} \), and \( d/dn \) denotes the difference derivative. A formula for the variables \( u_n \) analogous to ours was first derived by Novikov [2]. In 1977 the author became aware of a similar paper of Mumford.

References


Received by the Editors 12 October 1976.