# SCHUR-WEYL DUALITY

QUANG DAO

Abstract. We will switch gears this week and talk about the relationship between irreducible representations of the symmetric group $S_k$ and irreducible finite-dimensional representations of the general linear groups $GL_n$. This is known as Schur-Weyl duality. Along the way, we will introduce some key ingredients in the proof such as the Lie algebra $\mathfrak{gl}_n$ and the Double Commutant Theorem. Schur-Weyl duality also gives rise to the Schur functor, which generalizes the constructions of the symmetric and exterior powers. We will comment on this generalization and work out some non-trivial cases by hand.

Contents



## 1. Some Background

In order to describe the theorems and their proofs, we need some terminology from module theory and Lie theory.

**Definition 1.1.** A **simple module** is a non-zero module with no non-zero proper submodule.

**Definition 1.2.** A **semi-simple module** is a module that can be written as a direct sum of simple module.

**Proposition 1.3.** Submodules and quotient modules of a semi-simple module is semi-simple.

**Remark.** In the language of representation theory, any representation of a finite group is a semi-simple module, and simple modules correspond to irreducible representations.

The only definition we need from Lie theory is that of a Lie algebra.

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**Definition 1.4.** Let $\mathfrak{g}$ be a vector space over a field $k$ and let $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a skew-symmetric bilinear map.

Then $(\mathfrak{g}, [,])$ is a **Lie algebra** if $[,]$ also satisfies the Jacobi identity:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0.$$

**Remark.** For any Lie group, there is a corresponding Lie algebra. The only case we need in this talk is the correspondence between $\mathrm{GL}(V)$ and $\mathfrak{gl}(V)$. Here, $\mathfrak{gl}(V)$ is $\mathrm{End}(V)$ as a set, and $[X, Y] = XY - YX$.

**Definition 1.5.** Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two Lie algebras. A homomorphism of Lie algebra is a map $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi([a, b]) = [\phi(a), \phi(b)]$.

**Definition 1.6.** A representation of a Lie algebra $\mathfrak{g}$ is a vector space $V$ with a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$.

**Definition 1.7.** Given two representations $V$ and $W$ of a Lie algebra $\mathfrak{g}$, the tensor product $V \otimes W$ is also a representation of $\mathfrak{g}$ by the formula:

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw \ \forall X \in \mathfrak{g}.$$

## 2. Plan of the Proof

Given any finite dimensional vector space $V$, consider the vector space $V^{\otimes n}$. We can view the space as a right $S_n$-module, where the action of $\sigma \in S_n$ is described by

$$(v_1 \otimes \ldots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$

$V^{\otimes n}$ is also a left $\mathrm{GL}(V)$-module with the action of $g \in \mathrm{GL}(V)$ as

$$g \cdot (v_1 \otimes \ldots \otimes v_n) = g(v_1) \otimes \ldots \otimes g(v_n).$$

We can easily see that these actions commute. Schur-Weyl duality asserts something stronger, namely that:

**Theorem 2.1.** The span of the image of $S_n$ and $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ are centralizers of each other.

From this result and the Double Commutant Theorem, which will be introduced and proved below, we obtain a decomposition of $V^{\otimes n}$ as a representation of $S_n \times \mathrm{GL}(V)$. More specifically,

**Theorem 2.2** (**Schur-Weyl Duality**). We have, as a representation of $S_n \times \mathrm{GL}(V)$, the decomposition:

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda V \tag{2.1}$$

where $V_\lambda$'s are all the irreducible representations of $S_n$, and $S_\lambda V \simeq \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ is either an irreducible representation of $\mathrm{GL}(V)$ or is zero.

## 3. Double Commutant Theorem

The key theorem that allows us to extract the decomposition from 2.1 is the following result in module theory:

**Theorem 3.1.** Given a finite dimensional vector space $V$, let $A$ be a semi-simple subalgebra of $\mathrm{End}(V)$, and $B = \mathrm{End}_A(V)$. Then:

(1) B is semi-simple.
(2) A=$\mathrm{End}_B(V)$.
(3) As a $A \otimes B$-module, we have a decomposition:

$$V \simeq \bigoplus_i U_i \otimes W_i$$

where $U_i$'s are all the simple modules of $A$, and each $W_i \simeq \mathrm{Hom}_A(U_i, V)$ is either a simple module of $B$ or zero. Furthermore, the nonzero $W_i$'s are all the simple modules of $B$.

*Proof.* Since $A$ is semi-simple, we have the decomposition (as $A$-modules):

$$V \simeq \bigoplus_i U_i \otimes \mathrm{Hom}_A(U_i, V) \tag{3.1}$$

where the action of $A$ on $U_i \otimes \mathrm{Hom}_A(U_i, V)$ is given by: $g \cdot (u \otimes w) = (g \cdot u) \otimes w$.

Here, each $U_i$ is a simple module of $A$. We will prove that each $W_i = \mathrm{Hom}_A(U_i, V)$ is also a simple module of $B$.

Indeed, let $W \subset W_i$ be a non-zero submodule. In order to prove that $W = W_i$, it suffices to prove that for any $f, f' \in W$, there exists $b \in B$ such that $b \cdot f = f'$.

Since $U_i$ is a simple $A$-module, any function $f \in \mathrm{Hom}_A(U_i, V)$ is determined by where it sends an arbitrary nonzero element $u \in U_i$. Let $f(u) = v$ and $f'(u) = v'$. Then define $T \in \mathrm{End}(V)$ such that $T(a \cdot u) = a \cdot u'$ if $a \cdot u \in Au$ and $T(v) = v$ otherwise. It's easy to verify that $T \in \mathrm{End}_A(V) = B$ and $T \cdot f = f'$, so we are done.

Now, we also have:

$$\begin{aligned} B &= \mathrm{End}_A(V) \\ &\simeq \mathrm{Hom}_A(\bigoplus_i U_i \otimes W_i, V) \\ &\simeq \mathrm{Hom}_A(\bigoplus_i W_i \otimes U_i, V) \\ &\simeq \bigoplus_i \mathrm{Hom}(W_i, \mathrm{Hom}_A(U_i, V)) \\ &\simeq \bigoplus_i \mathrm{Hom}(W_i, W_i) \\ &\simeq \bigoplus_i \mathrm{End}(W_i) \end{aligned}$$

so from Artin-Wedderburn, it follows that $W_i$'s are all the simple modules of $B$. This also means that $B$ is semi-simple.

Notice that $U_i \simeq \mathrm{Hom}_B(W_i, V)$ because of the isomorphism $u \mapsto \mathrm{ev}_u$, where $\mathrm{ev}_u : \mathrm{Hom}_A(U_i, V) \to V$ is defined by $\mathrm{ev}_u(f) = f(u)$. Thus, we get these isomorphisms:

$$\begin{aligned}
\mathrm{End}_B(U) &= \mathrm{Hom}_B(\bigoplus_i U_i \otimes W_i, V) \\
&\simeq \bigoplus_i \mathrm{Hom}(U_i, \mathrm{Hom}_B(W_i, U)) \\
&\simeq \bigoplus_i \mathrm{Hom}(U_i, U_i) \\
&\simeq \bigoplus_i \mathrm{End}(U_i) \\
&\simeq A
\end{aligned}$$

where the last isomorphism is again due to Artin-Wedderburn. This establishes (2).

Finally, we can write (3.1) as:

$$V \simeq \bigoplus_i W_i \otimes \mathrm{Hom}_B(W_i, V)$$

so that the decomposition is also a $B$-module isomorphism. Hence, it's a $A \otimes B$-module isomorphism, which is (3). □

## 4. Proof of Schur-Weyl Duality

The proof of 2.1 comes in two steps; we will prove that the span of the image of $\mathfrak{gl}(V)$ and $S_n$ in $\mathrm{End}(V^{\otimes n})$ are centralizers of each other, then prove that the span of the image of $\mathfrak{gl}(V)$ and $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ are the same.

Thus, the proof of Schur-Weyl duality will be split into two theorems. The first theorem is the following:

**Theorem 4.1.** The subalgebra of $\mathrm{End}(V^{\otimes n})$ spanned by the image of $\mathfrak{gl}(V)$ is $B = \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n})$.

*Proof.* Recall that the action of $X \in \mathfrak{gl}(V)$ on $V^{\otimes n}$ is:

$$X \cdot (v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n} v_1 \otimes \ldots \otimes X \cdot v_i \otimes \ldots \otimes v_n.$$

This means that the image of $X$ in $\mathrm{End}(V^{\otimes n})$ is

$$\Pi_n(X) = \sum_{i=1}^{n} \mathrm{id} \otimes \ldots \otimes X \otimes \ldots \otimes \mathrm{id}$$

. This image obviously commute with any $\sigma \in S_n$, hence is in $B$.

On the other hand,

$$\begin{aligned}
B &= \mathrm{End}_{\mathbb{C}[S_n]}(V^{\otimes n}) \\
&= \mathrm{End}(V^{\otimes n})^{S_n} \\
&\simeq (End(V)^{\otimes n})^{S_n} \\
&\simeq \mathrm{Sym}^n \mathrm{End}(V)
\end{aligned}$$

which is spanned by $\{X^{\otimes n} \mid X \in \mathrm{End}(V)\}$.

However, we know from the theory of elementary polynomial that

$$X^{\otimes n} = \mathrm{P}(\Pi_n(X), \Pi_n(X^2), \ldots, \Pi_n(X^n))$$

for some polynomial P.

Thus, $X^{\otimes n}$ is in the span of the image of $\mathfrak{gl}(V)$ for any $X \in \mathrm{End}(V)$, and so $B$ is precisely the span of the image of $\mathfrak{gl}(V)$. □

The other part of the proof relates the image of $\mathfrak{gl}(V)$ and $\mathrm{GL}(V)$:

**Theorem 4.2.** The span of the image of $\mathfrak{gl}(V)$ and $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ are the same.

*Proof.* Let $B'$ the span of the image of $\mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$. From our discussion earlier that the action of $\mathrm{GL}(V)$ and $S_n$ on $V^{\otimes n}$ commutes, we get that $B' \subset B$.

Notice that the image of $g \in \mathrm{GL}(V)$ in $\mathrm{End}(V^{\otimes n})$ is $g^{\otimes n}$. Hence, to establish the reverse inclusion, the span of $\{g^{\otimes n} \mid g \in \mathrm{GL}(V)\}$ equal the span of $\{X^{\otimes n} \mid X \in \mathrm{End}(V)\}$.

Equivalently, it suffices to prove that any $X \in \mathrm{End}(V)$ is in the span of $\{g \mid g \in \mathrm{GL}(V)\}$. But this is not hard to prove, since there exists infinitely many $t \in \mathbb{R}$ such that $X + tI$ is invertible (hence in $\mathrm{GL}(V)$), and then $X = (X + tI) - tI$ is in the span of $\{g \mid g \in \mathrm{GL}(V)\}$. □

To finish the proof of Schur-Weyl duality, note that since $S_n$ is a finite group, the subalgebra spanned by the image of $S_n$ in $\mathrm{End}(V^{\otimes n})$ is semi-simple. From there, we can apply the Double Commutant Theorem to get the decomposition:

$$V^{\otimes n} \simeq \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda V.$$

## 5. Schur Functor and Examples

In the decomposition of Schur-Weyl Duality, we note that there is a map $V \mapsto S_\lambda V$ for a given tableau $\lambda$ of $n$. This can be upgraded to a functor $S_\lambda$ (so that any map $f : V \to W$ induces a map $S_\lambda f : S_\lambda V \to S_\lambda W$), called the **Schur functor**.

We can describe the space $S_\lambda V \simeq \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n})$ more explicitly using the definition of a Young symmetrizer.

**Definition 5.1.** Given a tableau $\lambda$ of $n$ with standard numbering, denote:

$$P_\lambda = \{g \in S_n \mid g \text{ preserves every row of } \lambda\},$$

and

$$Q_\lambda = \{g \in S_n \mid g \text{ preserves every column of } \lambda\}.$$

Furthermore, let

$$a_\lambda = \sum_{g \in P_\lambda} g \text{ and } b_\lambda = \sum_{g \in Q_\lambda} \mathrm{sgn}(g) g.$$

Then $c_\lambda = a_\lambda b_\lambda$ is called the **Young symmetrizer** of $\lambda$.

A classical result in the representation of the symmetric group states that these Young symmetrizers are in correspondence with the irreducible representations of $S_n$.

**Theorem 5.2.** Given any standard tableau $\lambda$ of $n$, the space $V_\lambda = \mathbb{C}[S_n]c_\lambda$ is an irreducible representation of $S_n$. Furthermore, any irreducible representation of $S_n$ is isomorphic to $\mathbb{C}[S_n]c_\lambda$ for some tableau $\lambda$.

From the theorem above, we can obtain a more explicit description of $S_\lambda V$. Since any representation of $S_n$ is self-dual, we have:

$$\begin{aligned} S_\lambda V &= \mathrm{Hom}_{S_n}(V_\lambda, V^{\otimes n}) \\ &\simeq (V_\lambda)^* \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\ &\simeq V_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\ &= \mathbb{C}[S_n]c_\lambda \otimes_{\mathbb{C}[S_n]} V^{\otimes n} \\ &\simeq V^{\otimes n} c_\lambda \end{aligned}$$

In general, there is no nice description for $S_\lambda V$, but special cases of $\lambda$ give some familiar constructions.

If $\lambda = (n)$, then

$$c_{(n)} = \sum_{g \in S_n} g, \text{ hence } S_{(n)}V \simeq \mathrm{Sym}^n V.$$

If $\lambda = (1, 1, \ldots, 1)$, then

$$c_{(1,1,\ldots,1)} = \sum_{g \in S_n} \mathrm{sgn}(g) g, \text{ hence } S_{(1,1,\ldots,1)}V \simeq \bigwedge^k V.$$

If $\lambda = (n-1, 1)$, then the description is not as nice, but we can still obtain that

$$S_{(n-1,1)}V \simeq \mathrm{Ker}(\mathrm{Sym}^{n-1}V \otimes V \to \mathrm{Sym}^n V),$$

and similarly,

$$S_{(2,1,\ldots,1)}V \simeq \mathrm{Ker}(\bigwedge^{n-1} V \otimes V \to \bigwedge^n V).$$

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