THE ELEMENTARY PROOF OF THE PRIME NUMBER THEOREM: AN HISTORICAL PERSPECTIVE

(by D. Goldfeld)

The study of the distribution of prime numbers has fascinated mathematicians since antiquity. It is only in modern times, however, that a precise asymptotic law for the number of primes in arbitrarily long intervals has been obtained. For a real number $x > 1$, let $\pi(x)$ denote the number of primes less than x. The prime number theorem is the assertion that

$$\lim_{x\to\infty} \pi(x) \Big/ \frac{x}{\log(x)} \; = \; 1.$$

This theorem was conjectured independently by Legendre and Gauss.

The approximation

$$\pi(x) = \frac{x}{A\log(x) + B}$$

was formulated by Legendre in 1798 **[Le1]** and made more precise in **[Le2]** where he provided the values $A = 1, B = -1.08366$. On August 4, 1823 (see **[La1]**, page 6) Abel, in a letter to Holmboe, characterizes the prime number theorem (referring to Legendre) as perhaps the most remarkable theorem in all mathematics.

Gauss, in his well known letter to the astronomer Encke, (see **[La1]**, page 37) written on Christmas eve 1849 remarks that his attention to the problem of finding an asymptotic formula for $\pi(x)$ dates back to 1792 or 1793 (when he was fifteen or sixteen), and at that time noticed that the density of primes in a chiliad (i.e. $[x, x + 1000]$) decreased approximately as $1/\log(x)$ leading to the approximation

$$\pi(x) \approx \mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}.$$

The remarkable part is the continuation of this letter, in which he said (referring to Legendre's $\frac{x}{\log(x)-A(x)}$ approximation and Legendre's value $A(x) = 1.08366$) that whether the quantity $A(x)$ tends to 1 or to a limit close to 1, he does not dare conjecture.

The first paper in which something was proved at all regarding the asymptotic distribution of primes was Tchebychef's first memoir (**[Tch1]**) which was read before the Imperial Academy of St. Petersburg in 1848. In that paper Tchebychef proved that if any approximation to $\pi(x)$ held to order $x/\log(x)^N$ (with some fixed large positive integer N) then that approximation had to be $\mathrm{Li}(x)$. It followed from this that Legendre's conjecture that $\lim\limits_{x\to\infty} A(x) = 1.08366$ was false, and that if the limit existed it had to be 1.

The first person to show that $\pi(x)$ has the order of magnitude $\frac{x}{\log(x)}$ was Tchebychef in 1852 **[Tch2]**. His argument was entirely elementary and made use of properties of factorials. It is easy to see that the highest power of a prime p which divides $x!$ (we assume x is an integer) is simply

$$\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots$$

where $[t]$ denotes the greatest integer less than or equal to t. It immediately follows that

$$x! = \prod_{p \le x} p^{[x/p]+[x/p^2]+\cdots}$$

and

$$\log(x!) = \sum_{p \le x} \left(\left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \left[\frac{x}{p^3}\right] + \cdots \right) \log(p).$$

Now $\log(x!)$ is asymptotic to $x\log(x)$ by Stirling's asymptotic formula, and, since squares, cubes, ... of primes are comparatively rare, and $[x/p]$ is almost the same as x/p, one may easily infer that

$$x \sum_{p \le x} \frac{\log(p)}{p} = x\log(x) \ + \ O(x)$$

from which one can deduce that $\pi(x)$ is of order $\frac{x}{\log(x)}$. This was essentially the method of Tchebychef, who actually proved that **[Tch2]**

$$B \ < \ \pi(x) \Big/ \frac{x}{\log(x)} \ < \ \frac{6B}{5}$$

for all sufficiently large numbers x, where

$$B = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30} \ \approx \ 0.92129$$

and

$$\frac{6B}{5} \ \approx \ 1.10555.$$

Unfortunately, however, he was unable to prove the prime number theorem itself this way, and the question remained as to whether an elementary proof of the prime number theorem could be found.

Over the years there were various improvements on Tchebychef's bound, and in 1892 Sylvester **[Syl1], [Syl2]** was able to show that

$$0.956 \ < \ \pi(x) \Big/ \frac{x}{\log(x)} \ < \ 1.045$$

for all sufficiently large x. We quote from Harold Diamond's excellent survey article **[D]**:

> The approach of Sylvester was *ad hoc* and computationally complex; it offered no hope of leading to a proof of the P.N.T. Indeed, Sylvester concluded in his article with the lament that "...we shall probably have to wait [for a proof of the P.N.T.] until someone is born into the world so far surpassing Tchebychef in insight and penetration as Tchebychef has proved himself superior in these qualities to the ordinary run of mankind."

The first proof of the prime number theorem was given by Hadamard **[H1]**, **[H2]** and de la Vallée Poussin **[VP]** in 1896. The proof was not elementary and made use of Hadamard's theory of integral functions applied to the Riemann zeta function $\zeta(s)$ which is defined by the absolutely convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

for $Re(s) > 1$. A second component of the proof was a simple trigonometric identity (actually, Hadamard used the doubling formula for the cosine function) applied in an extremely clever manner to show that the zeta function didn't vanish on the line $Re(s) = 1$. Later, several simplified proofs were given, in particular by Landau **[L]** and Wiener **[W1]**, **[W2]**, which avoided the Hadamard theory.

In 1921 Hardy (see **[B]**) delivered a lecture to the Mathematical Society of Copenhagen. He asked:

> "No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. Now we know that the theorem is roughly equivalent to a theorem about an analytic function, the theorem that Riemann's zeta function has no roots on a certain line. A proof of such a theorem, not fundamentally dependent on the theory of functions, seems to me extraordinarily unlikely. It is rash to assert that a mathematical theorem *cannot* be proved in a particular way; but one thing seems quite clear. We have certain views about the logic of the theory; we think that some theorems, as we say 'lie deep' and others nearer to the surface. If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten."

In the year 1948 the mathematical world was stunned when Paul Erdős announced that he and Atle Selberg had found a truly elementary proof of the prime number theorem which used only the simplest properties of the logarithm function. Unfortunately, this announcement and subsequent events led to a bitter dispute between these two mathematicians. The actual details of what transpired in 1948 have become distorted over time. A short paper, "The elementary proof of the prime number theorem," by E.G. Straus has been circulating for many years and has been the basis for numerous assertions over what actually happened. In 1987 I wrote a letter to the editors of the Atlantic Monthly (which was published) in response to an article about Erdős **[Ho]** which discussed the history of the elementary proof of the prime number theorem. At that time Selberg sent me his file of documents and letters (this is now part of **[G]**). Having been a close and personal friend of Erdős and also Selberg, having heard both sides of the story, and finally having a large collection of letters and documents in hand, I felt the time had come to simply present the facts of the matter with supporting documentation.

Let me begin by noting that in 1949, with regard to Paul Erdős's paper, "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem," the Bulletin of the American Mathematical Society informed Erdős that the referee does not recommend the paper for publication. Erdős immediately withdrew the paper and had it published in the Proceedings of the National Academy of Sciences [**E**] . At the same time Atle Selberg published his paper, "An elementary proof of the prime–number theorem," in the Annals of Mathematics [**S**]. These papers were brilliantly reviewed by A.E. Ingham [**I**].

The elementary proof of the prime number theorem was quite a sensation at the time. For his work on the elementary proof of the PNT, the zeros of the Riemann zeta function (showing that a positive proportion lie on the line $\frac{1}{2}$), and the development of the Selberg sieve method, Selberg received the Fields Medal [**B**] in 1950. Erdős received the Cole Prize in 1952 [**C**]. The Selberg sieve method, a cornerstone in elementary number theory, is the basis for Chen's [**Ch**] spectacular proof that every positive even integer is the sum of a prime and a number having at most two prime factors. Selberg is now recognized as one of the leading mathematicians of this century for his introduction of spectral theory into number theory culminating in his discovery of the trace formula [**A-B-G**] which classifies all arithmetic zeta functions. Erdős has also left an indelible mark on mathematics. His work provided the foundations for graph and hypergraph theory [**C–G**] and the probabilistic method [**A–S**] with applications in combinatorics and elementary number theory. At his death in 1996 he had more than 1500 published papers with many coauthored papers yet to appear. It is clear that he has founded a unique school of mathematical research, international in scope, and highly visible to the world at large.

Acknowledgment: The author would like to thank Enrico Bombieri, Melvyn Nathanson, and Atle Selberg for many clarifying discussions on historical detail. In addition I received a wide variety of helpful comments from Michael Anshel, Harold Diamond, Ron Graham, Dennis Hejhal, Jeff Lagarias, Attila Mate, Janos Pach, and Carl Pomerance.

March 1948: Let $\vartheta(x) = \sum_{p \le x} \log(p)$ denote the sum over primes $p \le x$. The prime number theorem is equivalent to the assertion that

$$\lim_{x \to \infty} \frac{\vartheta(x)}{x} = 1.$$

In March 1948 Selberg proved the asymptotic formula

$$\vartheta(x)\log(x) + \sum_{p \le x} \log(p)\vartheta\left(\frac{x}{p}\right) = 2x\log(x) + O(x).$$

He called this the fundamental formula.

We quote from Erdős's paper, "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem," Proc. Nat. Acad. Scis. 1949:

> "Selberg proved some months ago the above asymptotic formula, ... the ingenious proof is completely elementary ... Thus it can be used as a starting point for elementary proofs of various theorems which previously seemed inaccessible by elementary methods."

Quote from Selberg: Letter to H. Weyl Sept. 16, 1948

> "I found the fundamental formula ... in March this year ... I had found a more complicated formula with similar properties still earlier."

April 1948: Recall that $\vartheta(x) = \sum_{p \le x} \log(p)$. Define

$$a \;=\; \liminf \frac{\vartheta(x)}{x}, \qquad\qquad A \;=\; \limsup \frac{\vartheta(x)}{x}.$$

Sylvester's estimates guarantee that

$$0.956 \le a \le A \le 1.045.$$

In his letter to H. Weyl, Sept. 16, 1948, Selberg writes:

> "I got rather early the result that $a + A = 2$,"

The proof that $a + A = 2$ is given as follows. Choose x large so that

$$\vartheta(x) = ax + o(x).$$

Then since $\vartheta(x) \le Ax + o(x)$ it follows from Selberg's fundamental formula that

$$ax \log(x) + \sum_{p \le x} A \frac{x}{p} \log(p) \ge 2x \log(x) + o(x \log(x)).$$

Using Tchebychef's result that

$$\sum_{p \le x} \frac{\log(p)}{p} \sim \log(x)$$

it is immediate that $a + A \ge 2$. On the other hand, we can choose x large so that

$$\vartheta(x) = Ax + o(x).$$

Then since $\vartheta(x) \ge ax + o(x)$ it immediately follows as before that

$$Ax \log(x) + \sum_{p \le x} a \frac{x}{p} \log(p) \le 2x \log(x) + o(x \log(x)),$$

from which we get $a + A \le 2$. Thus

$$a + A = 2.$$

Remark: Selberg was aware of the fact (already in April 1948) that $a + A = 2$, and that the prime number theorem would immediately follow if one could prove either $a = 1$ or $A = 1$.

May–July 1948: We again quote from Selberg's letter to H. Weyl of Sept. 16, 1948.

> "In May I wrote down a sketch to the paper on Dirichlet's theorem, during June I did nothing except preparations to the trip to Canada. Then around the beginning of July, Turán asked me if I could give him my notes on the Dirichlet theorem so he could see it, he was going away soon, and probably would have left when I returned from Canada. I not only agreed to do this, but as I felt very much attached to Turán I spent some days going through the proof with him. In this connection I mentioned the *fundamental formula* to him, . . . However, I did not tell him the proof of the formula, nor about the consequences it might have and my ideas in this connection... I then left for Canada and returned after 9 days just as Turán was leaving. It turned out that Turán had given a seminar on my proof of the Dirichlet theorem where Erdős, Chowla, and Straus had been present, I had of course no objection to this, since it concerned something that was already finished from my side, though it was not published. In connection with this Turán had also mentioned, at least to Erdős, the *fundamental formula*, this I don't object to either, since I had not asked him not to tell this further."

July 1948: Quote from E.G. Straus' paper, "The elementary proof of the prime number theorem."

> "Turán who was eager to catch up with the mathematical developments that had happened during the war, talked with Selberg about his sieve method and now famous inequality (Fundamental Formula). He tried to talk Selberg into giving a seminar ... Selberg suggested Turán give the seminar.
>
> This Turán did for a small group of us, including Chowla, Erdős and myself, ... After the lecture ... there followed a brief discussion of the unexpected power of Selberg's inequality."
>
> "Erdős said,
>
> *I think you can also derive*
>
> $$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1$$
>
> *from this inequality.*
>
> In any case within an hour or two Erdős had discovered an ingenious derivation from Selberg's inequality. After presenting an outline of the proof to the Turán Seminar group, Erdős met Selberg in the hall and told him he could derive $\frac{p_{n+1}}{p_n} \to 1$ from Selberg's inequality."
>
> "Selberg responded something like this:

You must have made a mistake because with this result I can get an elementary proof of the prime number theorem, and I have convinced myself that my inequality is not powerful enough for that."

Quote from Weyl's letter to Selberg August 31, 1948

> "Is it not true that you were in possession of what Erdős calls the fundamental inequality and of the equation $a + A = 2$ for several months but could not prove $a = A = 1$ until Erdős deduced $\frac{p_{n+1}}{p_n} \to 1$ from your inequality?"

Here is Selberg's response in his letter to Weyl, Sept. 16, 1948.

> "Turán had mentioned to Erdős after my return from Montreal he told me he was trying to prove $\frac{p_{n+1}}{p_n} \to 1$ from my formula.
>
> Actually, I didn't like that somebody else started working on my unpublished results before I considered myself through with them."
>
> "But though I felt rather unhappy about the situation, I didn't say anything since after all Erdős was trying to do something different from what I was interested in.
>
> In spite of this, I became ... rather concerned that Erdős was working on these things . . .
> I, therefore, started very feverishly to work on my own ideas. On Friday evening Erdős had his proof ready (that $\frac{p_{n+1}}{p_n} \to 1$) and he told it to me.
>
> On Sunday afternoon I got my first proof of the prime number theorem.
> I was rather unsatisfied with the first proof because it was long and indirect. After a few days (my wife says two) I succeeded in giving a different proof."

Quote from Erdős's paper, "On a new method in elementary number theory which leads to an elementary proof of the prime number theorem," Proc. Nat. Acad. Scis. 1949:

> " Using (1) (fundamental formula) I proved that $\frac{p_{n+1}}{p_n} \to 1$ as $n \to \infty$. In fact, I proved the following slightly stronger result: To every ϵ there exists a positive $\delta(\epsilon)$ so that for x sufficiently large we have
>
> $$\pi\big(x(1+\epsilon)\big) - \pi(x) > \delta(\epsilon)x/\log(x)$$
>
> where $\pi(x)$ is the number of primes not exceeding x.
>
> I communicated this proof to Selberg, who, two days later . . . deduced the prime number theorem."

Recently, Selberg sent me a letter which more precisely specifies the actual dates of events.

Quote from Selberg's letter to D. Goldfeld, January 6, 1998:

> "July 14, 1948 was a Wednesday, and on Thursday, July 15 I met Erdős and heard that he was trying to prove $\frac{p_{n+1}}{p_n} \to 1$. I believe Turán left the

> next day (Friday, July 16), at any rate whatever lecture he had given (and I had not asked him to give one!) he had given before my return, and he was not present nor played any part in later events. Friday evening or it may have been Saturday morning, Erdős had his proof ready and told me about it. Sunday afternoon (July 18) I used his result (which was stronger than just $\frac{p_{n+1}}{p_n} \to 1$, he had proved that between x and $x(1+\delta)$ there are more than $c(\delta)\frac{x}{\log(x)}$ primes for $x > x_0(\delta)$, the weaker result would not have been sufficient for me) to get my first proof of the PNT. I told Erdős about it the next morning (Monday, July 19). He then suggested that we should talk about it that evening in the seminar room in Fuld Hall (as I thought, to a small informal group of Chowla, Straus and a few others who might be interested)."

In the same letter Selberg goes on to dispute Straus' recollection of the events.

> "Turán's lecture (probably a quite informal thing considering the small group) could not have been later than July 14, since it was before my return. Straus has speeded up events; Erdős told me he was trying to prove $\frac{p_{n+1}}{p_n} \to 1$ on July 15. He told me he had a proof only late on July 16 or possibly earlier the next day. Straus' quote is also clearly wrong for the following reasons; first, I needed more than just $\frac{p_{n+1}}{p_n} \to 1$ for my first proof of the PNT, second, I only saw how to do it on Sunday, July 18.
>
> It is true, however, as Erdős' and Straus' stories indicate, that when I first was told by Erdős that he was trying to prove $\frac{p_{n+1}}{p_n} \to 1$ from my formula, I tried to discourage him, by saying that I doubted whether the formula alone implied these things. I also said I had constructed a counterexample showing that the relation in the form
>
> $$f(x)\log x + \int_1^x f\left(\frac{x}{t}\right) df(t) = 2x\log x + (O(x))$$
>
> by itself does not imply that $f(x) \sim x$. It was true, I did have such an example. What I neglected to tell that in this example $f(x)$ (though positive and tending to infinity with x) was not monotonic! This conversation took place either in the corridor of Fuld Hall or just outside Fuld Hall so without access to a blackboard. This attempt to throw Erdős off the track (clearly not succeeding!) is somewhat understandable given my mood at the time.

Quote from Selberg's Paper, "An elementary proof of the prime–number theorem," Annals Math. 1949

> "From the Fundamental Inequality there are several ways to deduce the prime number theorem ... The original proof made use of the following result of Erdős $\frac{p_{n+1}}{p_n} \to 1$. Erdős's result was obtained entirely independent of my work."

Selberg's first proof that the prime number theorem followed from the fundamental formula is given both in [**E**] and [**S**]. The crux of the matter goes something like this. We may write the fundamental formula in the form

$$\frac{\vartheta(x)}{x} + \sum_{p\le x} \frac{\vartheta(x/p)}{x/p}\frac{\log(p)}{p\log(x)} = 2 + O\left(\frac{1}{\log(x)}\right).$$

Recall that a and A are the limit inferior and limit superior, respectively, of $\frac{\vartheta(x)}{x}$.

Now, choose x large so that $\frac{\vartheta(x)}{x}$ is near A. Since $a + A = 2$, it follows from the fundamental formula and

$$\sum_{p\le x} \frac{\log(p)}{p\log(x)} \sim 1,$$

that $\frac{\vartheta(x/p)}{x/p}$ must be near a for most primes $p \le x$. If $\mathcal{S}$ denotes the set of exceptional primes, then we have

$$\sum_{\substack{p\le x\\ p\in\mathcal{S}}} \frac{\log(p)}{p} \Bigg/ \sum_{p\le x} \frac{\log(p)}{p} \approx 0.$$

Now, choose a small prime $q \notin \mathcal{S}$ such that $\frac{\vartheta(x/q)}{x/q}$ is near a. Rewriting the fundamental formula with x replaced by x/q, the same argument as above leads one to conclude that $\frac{\vartheta(x/pq)}{x/pq}$ is near A for most primes $p \le x/q$. It follows that $\vartheta(x/p) \approx ax/p$ for most primes $p \le x$ and that $\vartheta(x/pq) \approx Ax/pq$ for most $p \le x/q$. A contradiction is obtained (using Erdős's idea of nonoverlapping intervals) unless $a = A = 1$.

The Erdős-Selberg dispute arose over the question of whether a joint paper (on the entire proof) or seperate papers (on each individual contribution) should appear on the elementary proof of the PNT.

August 20, 1948: Quote from a letter of Selberg to Erdős.

> "What I propose is the only fair thing: each of us can publish what he has actually done and get the credit for that, and not for what the other has done.
>
> You proved that
>
> $$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1.$$
>
> I would never have dreamed of forcing you to write a joint paper on this in spite of the fact that the essential thing in the proof of the result was mine."
>
> "Since there can be no reason for a joint paper, I am going to publish my proof as it now is. I have the opinion, . . . that I do you full justice by telling in the paper that my original proof depended on your result.

In addition to this I offered you to withhold my proof so your theorem could be published earlier (of course then without mentioning PNT).

I still offer you this. . .

If you don't accept this I publish my proof anyway."

Sept. 16, 1948: Quote from Selberg's letter to Weyl.

"when I came to Syracuse I discovered gradually through various sources that there had been made quite a publicity around the proof of the PNT. I have myself actually mentioned it only in one letter to one of my brothers ...

Almost all the people whom the news had reached seemed to attribute the proof entirely or at least essentially to Erdős, this was even the case with people who knew my name and previous work quite well."

Quote from E.G. Straus' , "The elementary proof of the prime number theorem."

"In fact I was told this story (I forget by whom) which may well not be true . . . When Selberg arrived in Syracuse he was met by a faculty member with the greeting: "

Have you heard the exciting news of what Erdős and some Scandinavian mathematician have just done? "

Quote from Selberg's letter to D. Goldfeld, January 6, 1998:

"This is not true. What I did hear shortly after my arrival were some reports (originating from the Boston–Cambridge area) where only Erdős was mentioned. Later there were more such reports from abroad."

Sept. 20, 1948: We quote from a second letter of Selberg to Erdős.

"I hope also that we will get some kind of agreement. But I cannot accept any agreement with a joint paper.

How about the following thing. You publish your result, I publish my newest proof, but with a satisfactory sketch of the ideas of the first proof in the introduction, and referring to your result. I could make a thorough sketch on 2 pages, I think, and this would not make the paper much longer. If you like, I could send you a sketch of the introduction.

I have thought to send my paper to the Annals of Math., they will certainly agree to take your paper earlier."

Sept. 27, 1948: Quote from Erdős's letter to Selberg.

"I have to state that when I started to work on $\frac{p_{n+1}}{p_n} \to 1$ you were very doubtful about success, in fact stated that you believe to be able to show

that the FUND. LEMMA does not imply the PNT (prime number theorem).
If you would have told me about what you know about a and A, I would have finished the proof of PNT on the spot.
Does it occur to you that if I would have kept the proof of $\frac{p_{n+1}}{p_n} \to 1$ to myself (as you did with $a + A = 2$) and continued to work on PNT . . . I would soon have succeeded and then your share of PNT would have only been the beautiful FUNDAMENTAL LEMMA.

Sept. 27, 1948: Quote from Erdős's letter to Selberg.

"I completely reject the idea of publishing only

$$\lim_{n\to\infty} \frac{p_{n+1}}{p_n} = 1.$$

and feel just as strongly as before that I am fully entitled to a joint paper. So if you insist on publishing your new proof all I can do is to publish our simplified proof, giving you of course full credit for your share (stating that you first obtained the PNT, using some of my ideas and my theorem).

Also, I will of course gladly submit the paper to Weyl first, if he is willing to take the trouble of seeing that I am scrupulously fair to you.

Quote from E.G. Straus: "The elementary proof of the prime number theorem."

"It was Weyl who caused the Annals to reject Erdős's paper and published only a version by Selberg which circumvented Erdős's contribution, without mentioning the vital part played by Erdős in the first elementary proof, or even the discovery of the fact that such a proof was possible."

Quote from Selberg's letter to D. Goldfeld, January 6, 1998:

"This is wrong on several points, my paper mentioned and sketched in some detail how Erdős's result played a part and was used in the first elementary proof of PNT, but that first proof was mine as surely as Erdoš' result was his. Also the discovery that such a proof was possible was surely mine. After all, you don't know that it is possible to prove something until you have done so!"

Excerpt: Handwritten Note by Erdős:

"It was agreed that Selberg's proof should be in the Annals of Math., mine in the Bulletin. Weyl was supposed to be the referee. To my great surprise Jacobson the referee. . . The Bulletin wrote that the referee does not recommend my paper for publication.

. . .

I immediately withdrew the paper and planned to publish it in the JLMS but . . . had it published in the Proc. Nat. Acad."

Feb. 15, 1949: Quote from H. Weyl's letter to Jacobson

> "I had questioned whether Erdős has the right to publish things which are admittedly Selberg's. . . I really think that Erdős's behavior is quite unreasonable, and if I were the responsible editor I think I would not be afraid of rejecting his paper in this form.
>
> But there is another aspect of the matter. It is probably not as easy as Erdős imagines to have his paper published in time in this country if the Bulletin rejects it. . . So it may be better to let Erdős have his way. No great harm can be done by that. Selberg may feel offended and protest (and that would be his right), but I am quite sure that the two papers – Selberg's and Erdős's together – will speak in unmistakable language, and that the one who has really done harm to himself will be Erdős."

Quote from E.G. Straus: "The elementary proof of the prime number theorem."

> "The elementary proof has so far not produced the exciting innovations in number theory that many of us expected to follow. So, what we witnessed in 1948, may in the course of time prove to have been a brilliant but somewhat incidental achievement without the historic significance it then appeared to have."

Quote from Selberg's letter to D. Goldfeld, January 6, 1998:

> "With this last quote from Straus, I am in agreement (actually I did not myself expect any revolution from this). The idea of the local sieve, however, has produced many things that have not been done by other methods."

Remark: To this date, there have been no results obtained from the elementary proof of the PNT that cannot be obtained in stronger form by other methods. Other elementary methods introduced by both Selberg and Erdős have, however, led to many important results in number theory not attainable by any other technique.

Dec. 4, 1997: Letter from Selberg to D. Goldfeld.

> "The material I have is nearly all from Herman Weyl's files, and was given to me probably in 1952 or 1953 as he was cleaning out much of his stuff in Princeton, taking some to Zurich and probably discarding some. The letters from Weyl to myself was all that I kept when I left Syracuse in 1949, all the rest I discarded. Thus there are gaps. Missing is my first letter to Erdős as well as his reply to it. . .
>
> I did not save anything except letters from Weyl because I was rather disgusted with the whole thing. I never lectured on the elementary proof of the PNT after the lecture in Syracuse, mentioned in the first letter to Herman Weyl. However, I did at Cornell U. early in 1949 and later at

an AMS meeting in Baltimore gave a lecture with an elementary proof of (using the notation of Beurling generalized primes & integers) the fact that if

$$N(x) = Ax + o\left(\frac{x}{\log^2(x)}\right)$$

then

$$\Pi(x) = \frac{x}{\log(x)} + o\left(\frac{x}{\log(x)}\right).$$

Beurling has the same conclusion if

$$N(x) = Ax + O\left(\frac{x}{(\log(x))^\alpha}\right),$$

with $\alpha > \frac{3}{2}$. I never published this.

Erdős of course lectured extensively in Amsterdam, Paris, and other places in Europe. After his lecture in Amsterdam, Oct. 30, 1948, v.d. Corput wrote up a paper, Scriptum 1, Mathematisch Centrum, which was the first published version!"

References

[A–B–G] K.E. Aubert, E. Bombieri, D. Goldfeld, *Number theory, trace formulas and discrete groups, symposium in honor of Atle Selberg*, Academic Press Inc. Boston (1989).

[A–S] N. Alon, J. Spencer, *The probabilistic method,* John Wiley & Sons Inc., New York (1992).

[B] H. Bohr, *Address of Professor Harold Bohr,* Proc. Internat. Congr. Math. (Cambridge, 1950) vol 1, Amer. Math. Soc., Providence, R.I., 1952, 127–134.

[Ch] J. Chen, *On the representation of a large even integer as the sum of a prime and the product of at most two primes,* Sci. Sinica **16** (1973), 157–176.

[C–G] F. Chung, R. Graham, *Erdős on graphs: his legacy of unsolved problems,* A.K. Peters, Ltd., Wellesley, Massachusetts (1998).

[C] L.W. Cohen, *The annual meeting of the society,* Bull. Amer. Math. Soc **58** (1952), 159–160.

[D] H.G. Diamond, *Elementary methods in the study of the distribution of prime numbers,* Bull. Amer. Math. Soc. vol. 7 number 3 (1982), 553–589.

[E] P. Erdős, *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,* Proc. Nat. Acad. Scis. U.S.A. **35** (1949), 374–384.

[G] D. Goldfeld, *The Erdős–Selberg dispute: file of letters and documents,* to appear.

[H1] J. Hadamard, *Étude sur les proprietés des fonctions entiéres et en particulier d'une fonction considérée par Riemann,* J. de Math. Pures Appl. (4) **9** (1893), 171–215; reprinted in Oeuvres de Jacques Hadamard, C.N.R.S., Paris, 1968, vol 1, 103–147.

[H2] J. Hadamard, *Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques,* Bull. Soc. Math. France **24** (1896), 199–220; reprinted in Oeuvres de Jacques Hadamard, C.N.R.S., Paris, 1968, vol 1, 189–210.

[Ho] P. Hoffman, *The man who loves only numbers,* The Atlantic, November (1987).

[I] A.E. Ingham, Review of the two papers: *An elementary proof of the prime–number theorem,* by A. Selberg and *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem,* by P. Erdős. Reviews in Number Theory as printed in Mathematical Reviews 1940-1972, Amer. Math. Soc. Providence, RI (1974). See N20-3, Vol. 4, 191–193.

[La1] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen,* Teubner, Leipzig (1909), 2 volumes, reprinted by Chelsea Publishing Company, New York (1953).

[La2] E. Landau, *Über den Wienerschen neuen Weg zum Primzahlsatz,* Sitzber. Preuss. Akad. Wiss., 1932, 514–521.

[Le1] A.M. Legendre, *Essai sur la théorie des nombres,* 1. Aufl. Paris (Duprat) (1798).

[Le2] A.M. Legendre, *Essai sur la théorie des nombres,* 2. Aufl. Paris (Courcier) (1808).

[S] A. Selberg, *An elementary proof of the prime–number theorem,* Ann. of Math. (2) **50** (1949), 305–313; reprinted in Atle Selberg Collected Papers, Springer–Verlag, Berlin Heidelberg New York, 1989, vol 1, 379–387.

[Syl1] J.J. Sylvester, *On Tchebycheff's theorem of the totality of prime numbers comprised within given limits,* Amer. J. Math. **4** (1881), 230–247.

[Syl2] J.J. Sylvester, *On arithmetical series,* Messenger of Math. (2) **21** (1892), 1–19 and 87–120.

[Tch1] P.L. Tchebychef, *Sur la fonction qui détermine la totalité des nombres premiers inférieurs à une limite donnée,* Mémoires présentés à l'Académie Impériale des Sciences de St.-Pétersbourg par divers Savants et lus dans ses Assemblées, Bd. 6, S. (1851), 141–157.

[Tch2] P.L. Tchebychef, *Mémoire sur les nombres premiers,* J. de Math. Pures Appl. (1) **17** (1852), 366–390; reprinted in Oeuvres **1** (1899), 49–70.

[VP] C.J. de la Vallée Poussin, *Recherches analytiques sur la théorie des nombres premiers,* Ann. Soc. Sci. Bruxelles **20** (1896), 183–256.

[W1] N. Wiener, *A new method in Tauberian theorems,* J. Math. Physics M.I.T. **7** (1927–28), 161–184.

[W2] N. Wiener, *Tauberian theorems,* Ann. of Math. (2) **33** (1932), 1–100.