Groups and Representations II: Problem Set 2

Due Monday, March 3

Problem 1:

Let $\mathfrak{g}_\alpha$ be the root-space corresponding to the root α. Prove that $\dim\, \mathfrak{g}_\alpha = 1$ and that the only other root proportional to α is $-\alpha$. Here $\mathfrak{g}$ is a complex Lie algebra.

Problem 2:

Consider the Lie algebra $\mathfrak{g}_\alpha = \mathfrak{sl}(n, \mathbf{C})$, the complexification of $\mathfrak{su}(n)$. For each root α_{jk} $(j \neq k)$, let

$$E_{\alpha_{jk}} = E_{jk} \in \mathfrak{g}_\alpha$$

and define

$$H_{\alpha_{jk}} = E_{jj} - E_{kk}$$

Show that the maps

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \to H_{\alpha_{jk}}$$

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to E_{\alpha_{jk}}$$

$$X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to E_{-\alpha_{jk}}$$

define a Lie algebra isomorphism from $\mathfrak{sl}(2, \mathbf{C})$ to the subalgebra of $\mathfrak{sl}(n, \mathbf{C})$ spanned by

$$H_{\alpha_{jk}}, E_{\alpha_{jk}}, E_{-\alpha_{jk}}$$

Problem 3:

Let α and β be non-proportional roots with an acute angle between them. Prove that $\alpha - \beta$ is also a root.

Problem 4:

Define

$$\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$$

where R_+ is the set of positive roots for a Lie algebra $\mathfrak{g}$. For every root α in a corresponding set S of simple roots, show that

$$s_\alpha(\delta) = \delta - \alpha$$

For every element w in the Weyl group, show that

$$w(\delta) = \delta - \sum_S n_\alpha \cdot \alpha$$

where n_α is an integer.

Problem 5:

There are four Dynkin diagrams with two nodes, corresponding to having zero, one, two or three lines joining the two nodes. From these Dynkin diagrams, construct the root systems of the four corresponding Lie algebras. Draw these root systems and the corresponding diagrams, identifying the simple roots, positive roots, Weyl chambers and fundamental Weyl chamber. Using the fact that reflecting the simple roots generates the Weyl group, identify the Weyl groups for these four cases. Compute the Cartan matrix for these four cases.