Groups and Representations II: Problem Set 1

Due Wednesday, February 12

Problem 1:

Consider the Lie algebra $\mathfrak{so}(3)$ of the group SO(3) of 3 by 3 orthogonal matrices and choose an explicit isomorphism

$$\mathbf{R}^3 \to \mathfrak{so}(3)$$

a) Show that under this isomorphism the Lie bracket $[\cdot, \cdot]$ gets identified with the vector product in $\mathbf{R}^3$.
b) Given a vector $\mathbf{x} \in \mathbf{R}^3$, find the matrix that represents $ad(\mathbf{x})$.
c) Characterize the one parameter subgroup of $SO(3)$ that corresponds to the Lie algebra element identified with $\mathbf{x}$. What sort of rotations are these?
e) Explicitly compute the Killing form as a bilinear form on $\mathbf{R}^3$.

Problem 2:

Show that the Killing form of a Lie algebra $\mathfrak{g}$ satisfies

$$K([X, Y], Z) = K(X, [Y, Z])$$

for all $X, Y, Z \in \mathfrak{g}$.

Problem 3:

For a Lie algebra with non-degenerate Killing form $K(\cdot, \cdot)$, the quadratic Casimir operator is defined as follows. Take any basis $X_1, \cdots, X_n$ of $\mathfrak{g}$ and the dual basis $X^1, \cdots, X^n$ with respect to K, (i.e. $K(X_i, X^j) = \delta_{ij}$, note all these are in $\mathfrak{g}$, not $\mathfrak{g}^*$). Define

$$C_2 = \sum_{i=1}^{n} X_i X^i$$

Show that C_2 is in the center $Z(\mathfrak{g})$ (i.e. $[X, C_2] = 0$ for all $X \in \mathfrak{g}$).

Problem 4:

For the Lie group $SL(2, \mathbf{R})$, explicitly identify the Lie algebra $\mathfrak{sl}(2, \mathbf{R})$ with $\mathbf{R}^3$ and
a) Explicitly construct the adjoint representations of $Ad(SL(2, \mathbf{R}))$ and $ad(\mathfrak{sl}(2, \mathbf{R}))$.
b) Explicitly construct the Killing form on $\mathfrak{sl}(2, \mathbf{R})$.
c) Use $Ad(SL(2, \mathbf{R}))$ to show that the $SL(2, \mathbf{R})$ is a double cover of the Lie group $SO(2, 1)$.

Problem 5:

Prove that the set of right-invariant vector fields on a Lie group G forms a Lie algebra under the Lie bracket and is isomorphic as a vector space to T_eG. Let

$\phi : G \to$ be the diffeomorphism of G defined by $\phi(g) = g^{-1}$. Show that if X is a left-invariant vector field on G, then $\phi_*(X)$ is the right-invariant vector field whose valued at e is $-X(e)$. Prove that

$$X \to \phi_*(X)$$

give a Lie algebra isomorphism of the Lie algebra of left-invariant vector fields on G with the Lie algebra of right-invariant vector fields on G.

Problem 6:

For X an n by n complex matrix, show that

$$\frac{Id - e^{-X}}{X}$$

is invertible iff X has no eigenvalue of the form $\lambda = 2\pi i n$, n a non-zero integer. *Remark*: Using our formula for the differential of the exponential map, this shows that the exponential map is a local diffeomorphism near $X \in \mathfrak{g}$ iff $ad(X)$ has no eigenvalue of the form $\lambda = 2\pi i n$.

Problem 7:

Use the Baker-Campbell-Hausdorff theorem we derived to get the explicit formula

$$ln(exp(X)exp(Y)) = X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}([X,[X,Y]]-[Y,[X,Y]])+\text{higher order terms}$$