Topics in Representation Theory: Roots and Complex Structures

1 More About Roots

To recap our story so far: we began by identifying an important abelian subgroup of G, the maximal torus T. By restriction any G representation π is a T representation $\pi_{|T}$. In general $\pi_{|T}$ is a reducible T representation and the irreducible representations of T (which are all one-dimensional) that occur in $\pi_{|T}$ are called the *weights* of π. Irreducible representations of T, and thus weights, are labelled by an element of $\mathfrak{t}^*$, one that is integral on the integer lattice $exp^{-1}(e) \subset T$. When we refer to "weights", we will often be referring to these labels. If π is represented on a vector space V, the weight-space V_α corresponding to a weight α will be the sum of the one-dimensional subspaces of V that are irreducible representations of T with weight α.

The following explanation of how the geometry of G/T is linked to representation theory is part of a much larger story. For more details, and much material on the relation of the cohomology of G/T to representation theory, see [2].

G acts by conjugation on itself

$$g \to hgh^{-1}$$

leaving the identity invariant. G also acts on the space of right cosets G/T by the left action

$$h \in G : gT \to hgT$$

The maximal torus T is a subgroup of G and so also acts by conjugation. This conjugation action gives an action on G/T which is identical to the left action on G/T by T since

$$g \to tgt^{-1}$$

induces on right cosets

$$gT \to tgT$$

Considering the differential of the conjugation action at the identity of G gives the adjoint representation Ad on $\mathfrak{g}$ and restricting this representation to T gives a reducible representation $Ad_{|T}$ on $\mathfrak{g}$. The trivial representation of T will occur with multiplicity $rank(G)$ and there will be some number of non-trivial weights α_i. These are called the roots of G. The decomposition of $\mathfrak{g}$ into weight-spaces is

$$\mathfrak{g} = \mathfrak{t} \oplus \sum_i g_{\alpha_i}$$

The root spaces g_{α_i} are two dimensional, note that this implies that $dim\ G - dim\ T$ is even dimensional and the coset space G/T is an even-dimensional manifold. We'll sometimes write

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}/\mathfrak{t}$$

note that $\mathfrak{t}$ is not an ideal in $\mathfrak{g}$ so the quotient is just a quotient of vector spaces not of Lie algebras. While $\mathfrak{t}$ is a Lie algebra, $\mathfrak{g}/\mathfrak{t}$ isn't.

The simplest example to keep in mind is $G = SU(2) = S^3$, where $T = U(1)$ and $G/T = S^2$. Here the Lie algebra is isomorphic to $\mathbf{R}^3$ with basis S_1, S_2, S_3. Choosing T to be the $U(1)$ subgroup you get from exponentiating S_3, the one dimensional vector space generated S_3 is $\mathfrak{t}$ and a trivial representation space for T. The two dimensional vector space generated by S_1 and S_2 is a nontrivial representation space for T.

2   Complex Structures

The even-dimensional real vector space $\mathfrak{g}/\mathfrak{t}$ decomposes as a sum of non-trivial two-dimensional real representations of T. Recall that we showed that these are labelled by non-zero integers, and that the representations corresponding to n and $-n$ are equivalent. These real representations of T correspond to one-dimensional complex representations, with the choice of $\pm n$ corresponding to the choice of $\mathbf{C}$ or its conjugate, the same space with i replaced by its conjugate $-i$. This choice of $\pm i$ is what we call the choice of a complex structure on the real plane. We want a square root of the transformation $-Id$ and can get this by a rotation by $\pi/2$, either clockwise or counter-clockwise.

In general given any even dimensional real vector space $\mathbf{R}^{2n}$, there are many ways we can turn it into a complex vector space and identify it with $\mathbf{C}^n$. To make it a vector space over the complex numbers we just need to know how $\sqrt{-1}$ acts on the vector space. We'll define

Definition 1. *A complex structure J on a real vector space V is an element $J \in GL(V)$ such that $J^2 = -1$.*

If V has an inner product, we can work with $J \in O(V)$, orthogonal complex structures, and will generally do this. The two components of $O(V)$ correspond to the choice of orientation of V since the choice of a complex structure fixes an orientation (any two bases of V of the form

$$X_1, \cdots, X_n, JX_1, \cdots, JX_n$$

are related by a transformation with positive determinant).

Given a complex structure J, V is a complex vector space with multiplication by a complex number $a + ib$ defined for $v \in V$ by

$$(a + ib)v = av + bJv$$

Another way of thinking about complex structures is to consider the complexification

$$V_{\mathbf{C}} = V \otimes_{\mathbf{R}} \mathbf{C}$$

and note that given complex structure J, $V_{\mathbf{C}}$ can be decomposed into $\pm i$ eigenspaces of J.

$$V_{\mathbf{C}} = V^{0,1} \oplus V^{1,0}$$

where J acts by multiplication by i on $V^{0,1}$, by $-i$ on $V^{1,0}$.

A manifold M is an almost-complex manifold if there is a complex structure J defined continuously on each tangent space. It is a complex manifold if this structure is integrable, i.e. the Lie bracket respects the $V^{0,1} \oplus V^{1,0}$ decomposition. This means that Lie brackets of vector fields with values in $V^{0,1}$ are in $V^{0,1}$. It turns out that for such a manifold we can locally choose coordinate charts that are $\mathbf{C}^n$, with transition functions between charts holomorphic functions.

We are interested here in the case $V = \mathfrak{g}/\mathfrak{t}$. We have seen that this is a real T representation and its weights (the roots) are non-zero and come with an ambiguity of sign. Choosing a complex structure on V gives a decomposition

$$(\mathfrak{g}/\mathfrak{t})_{\mathbf{C}} = (\mathfrak{g}/\mathfrak{t})^{0,1} \oplus (\mathfrak{g}/\mathfrak{t})^{1,0}$$

Complexification changes our real T representation into a pair of complex T representations with weights of opposite sign. Making some choice of which roots are "positive", we have

$$(\mathfrak{g}/\mathfrak{t})_{\mathbf{C}} = \sum_{+\text{roots } \alpha_i} (g_{\alpha_i} \oplus g_{-\alpha_i})$$

It turns out that the Lie bracket of two positive roots is positive, so the almost complex structure on G/T defined by the decomposition into positive and negative roots is integrable and gives G/T a complex structure. In our simple example $G = SU(2)$, G/T is the sphere, which is definitely a complex manifold, the Riemann sphere.

Note that the choice of decomposition into positive and negative roots (and thus the choice of complex structure) is arbitrary. Recall that the Weyl group acts as automorphisms of T and thus acts on the space of weights. It will as well act on the set of possible G-invariant complex structures on G/T. In the simplest example $G = SU(2)$, the non-trivial element of the Weyl group takes the standard complex structure on the Riemann sphere $S^2 = G/T$ to the conjugate one.

3 The Complexification of the Lie Algebra

So far we have been considering the action of T on $\mathfrak{g}$, now we will differentiate and consider the adjoint action of $\mathfrak{t}$ on $\mathfrak{g}$. This can be given rather explicitly in terms of commutators. Since T is abelian

$$[\mathfrak{t}, \mathfrak{t}] = 0$$

i.e. all commutators of elements of $\mathfrak{t}$ with each other are zero.

We want to also study the commutators of $\mathfrak{t}$ with the root spaces that make up $\mathfrak{g}/\mathfrak{t}$, the rest of $\mathfrak{g}$. We have seen that in order to identify the root spaces with complex representations of T, we need to complexify $\mathfrak{g}/\mathfrak{t}$ and choose a complex structure.

The standard thing to do at this point is to also complexify $\mathfrak{t}$, then the roots are the eigenvalues of the adjoint $\mathfrak{t}_{\mathbf{C}}$ action on $(\mathfrak{g}/\mathfrak{t})_{\mathbf{C}}$.

Another approach to this subject is to from the beginning work in terms of the complexified Lie algebra $\mathfrak{g}_C$ which can now be expressed as

$$\mathfrak{g}_C = \mathfrak{t}_{\mathbf{C}} \oplus \sum_{+\text{roots } \alpha_i} (g_{\alpha_i} \oplus g_{-\alpha_i})$$

One of the more confusing aspects of this subject is that in the passage from the real to the complex Lie algebra

$$\mathfrak{t} \to \mathfrak{t}_{\mathbf{C}}$$

what people generally do is actually not take $\mathfrak{t}$ to be the real elements of $\mathfrak{t}_{\mathbf{C}}$, but instead multiply $\mathfrak{t}$ by $2\pi i$ so the elements of $\mathfrak{t}$ sit inside $\mathfrak{t}_{\mathbf{C}}$ as purely imaginary elements, rescaled by 2π. For the case of T a circle, this is a change of variable from x, taking values from 0 to 1 as you go around T, to $u = 2\pi i x$, taking values from 0 to $= 2\pi i$ as you go around the group. Some authors such as [1] refer to the roots we have been using so far as the "real roots" and roots defined in terms of $\mathfrak{t}_{\mathbf{C}}$ as "complex roots".

The different definitions are:

Definition 2 (Real Roots). *Real roots are labelled by maps*

$$\alpha : \mathfrak{t} \to \mathbf{R}$$

integral on $\exp^{-1}(e) \subset T$. The two-dimensional subspace of $\mathfrak{g}/\mathfrak{t}$ on which T acts irreducibly with weight corresponding to this root is called the root space $\mathfrak{g}_\alpha$. On this root space $\mathfrak{g}_\alpha$, the adjoint action of T is given explicitly by

$$Ad([x_1], \cdots, [x_k])_{|\mathfrak{g}_\alpha} = \begin{pmatrix} \cos(2\pi\alpha(x_1, \cdots, x_k)) & -\sin(2\pi\alpha(x_1, \cdots, x_k)) \\ \sin(2\pi\alpha(x_1, \cdots, x_k)) & \cos(2\pi\alpha(x_1, \cdots, x_k)) \end{pmatrix}$$

The adjoint action of the Lie algebra $\mathfrak{t}$ on $\mathfrak{g}_\alpha$ is given by

$$ad(x_1, \cdots, x_k)_{|\mathfrak{g}_\alpha} = \begin{pmatrix} 0 & -2\pi\alpha(x_1, \cdots, x_k) \\ 2\pi\alpha(x_1, \cdots, x_k) & 0 \end{pmatrix}$$

The eigenvalues of this last matrix are $\pm 2\pi i\alpha$ and complexifying allows us to use root spaces that are eigenvectors with these eigenvalues. We can define complex roots as follows:

Definition 3 (Complex Roots). *Complex roots are labelled by complex linear maps*

$$\alpha_c : \mathfrak{t}_{\mathbf{C}} \to \mathbf{C}$$

4

The one-dimensional complex subspace on which T acts irreducibly with weight corresponding to this root is called the root space $\mathfrak{g}_{\alpha_c}$ On this root space $\mathfrak{g}_{\alpha_c}$ the adjoint action of T is given explicitly by

$$Ad([u_1], \cdots, [u_k])_{|\mathfrak{g}_{\alpha_c}} = e^{\alpha_c(u_1, \cdots, u_k)}$$

The adjoint action of the Lie algebra $\mathfrak{t}_{\mathbf{C}}$ on $\mathfrak{g}_{\alpha_c}$ is given by

$$ad(u_1, \cdots, u_k)_{|\mathfrak{g}_{\alpha_c}} = \alpha_c(u_1, \cdots, u_k)$$

In other words, in terms of complex roots we have the commutation relations

$$[H, X] = \alpha_c(H)X$$

for $H \in \mathfrak{t}_{\mathbf{C}}$ and $X \in \mathfrak{g}_{\alpha_c}$.

The rest of the commutation relations can now be derived from the following, for simplicity we'll write this in terms of complex roots, although I've dropped the subscripts indicating this.

Lemma 1.

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

Proof:
If $H \in \mathfrak{t}, X_\alpha \in \mathfrak{g}_\alpha, X_\beta \in \mathfrak{g}_\beta$ the Jacobi identity gives

$$
\begin{aligned}
[H, [X_\alpha, X_\beta]] &= -[X_\alpha, [X_\beta, H]] - [X_\beta, [H, X_\alpha]] \\
&= [X_\alpha, \beta(H)X_\beta] - [X_\beta, \alpha(H)X_\alpha] \\
&= (\alpha(H) + \beta(H))[X_\alpha, X_\beta]
\end{aligned}
$$

Note that this shows that commutators of elements in positive root spaces are in a positive root space.

4 An Example: $G = SU(2)$

For the case $G = SU(2)$, $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathfrak{g}_{\mathbf{C}} = \mathfrak{sl}(2, \mathbf{C})$. Using the Pauli matrices σ_i we can explicitly represent element of $\mathfrak{sl}(2, \mathbf{C})$ as follows

$$\mathfrak{sl}(2, \mathbf{C}) = \{u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3, \quad u_i \in \mathbf{C}\}$$

and the standard choice of a maximal torus $T = U(1) \subset SU(2)$ leads to

$$\mathfrak{t}_{\mathbf{C}} = \{u_3\sigma_3, \quad u_3 \in \mathbf{C}\}$$

The eigenvectors of $H = u_3\sigma_3$ acting on $(\mathfrak{g}/\mathfrak{t})_{\mathbf{C}}$ are

$$X^{\pm} = \sigma_1 \pm i\sigma_2$$

and one finds

$$[u_3\sigma_3, X^+] = 2u_3 X^+$$
$$[u_3\sigma_3, X^-] = -2u_3 X^-$$

One thus has two complex roots of opposite sign

$$\alpha_c(u) = \pm 2u$$

References

[1] Bröcker, T. and tom Dieck, T.,*Representations of Compact Lie Groups,* Springer-Verlag, 1985.

[2] Borel, A., and Hirzebruch, F., Characteristic classes and homogeneous spaces, I, Amer. J. Math. **80** (1958), 458-538.