Topics in Representation Theory: Maximal Tori and the Weyl Group

The next part of this course will be concerned with compact, connected Lie groups and their representations. Our goal is to

- Classify compact connected Lie groups
- Classify all irreducible representations of such groups
- Calculate the characters of these irreducible representations

Recall that the Peter-Weyl theorem tells one that the matrix elements of irreducible representations form an orthonormal basis of $L^2(G)$ for G a compact Lie group. Equivalently, there is an isomorphism of Hilbert spaces

$$L^2(G) = \oplus_{\pi \in \hat{G}} V_\pi \otimes V_\pi^*$$

where the Hilbert space direct sum is over all irreducible representations of G. Under the left action of G, the infinite dimensional representation on $L^2(G)$ breaks up into finite dimensional representation spaces V_π with multiplicity $dim\ V_\pi$. What the Peter-Weyl theorem doesn't do is tell us what the representations are and how to tell them apart. We would like to know how to project from $L^2(G)$ onto each of the $V_\pi \otimes V_\pi^*$. This requires classifying the irreducible representations and finding their characters.

1 Maximal Tori

For the case of $G = U(1)$, Fourier analysis of functions on the circle tells us that irreducible representations are all one dimensional, labelled by an integer n and have character

$$\chi(e^{i\theta}) = e^{in\theta}$$

A group consisting of a finite product of k copies of $U(1)$ will be called a torus and its irreducible representations are labelled by k integers. For compact, connected G, we will approach the problem of determining its irreducible representations by considering a torus subgroup T of G that is as large as possible and relating the representation theory of G (which we don't understand) and that of T (which we do understand). This will involve the space G/T of T cosets of G. The geometry and topology of G/T is quite interesting and intimately involved in the problem of finding the representations of G.

The spaces G/T have several different characterizations that we will explore later. They are not only Kähler manifolds, but projective algebraic varieties and so can be studied with the methods of algebraic geometry. They are sometimes called "flag manifolds", since in the case of $G = U(n)$, G/T is the space of complex "flags" in $\mathbf{C}^n$, i.e chains of inclusions

$$\mathbf{C} \subset \mathbf{C}^2 \subset \cdots \subset \mathbf{C}^n$$

Another important way of characterizing these spaces is as orbits of elements in $\mathfrak{g}$ under the adjoint action of G. They are symplectic manifolds and can be thought of as possible phase spaces of classical mechanical systems, systems whose quantization leads to a quantum theory with finite dimensional Hilbert space given by a representation of G.

Topologically, the cohomology ring of G/T turns out to be quite simple to describe, with all cohomology in even degrees.

We'll begin with the following definition:

Definition 1 (Maximal Torus). *A torus T of G is a subgroup of G isomorphic to a product of $U(1)$ factors. A maximal torus T is a torus such that there is no torus H in G such that*

$$T \subset H$$

is a proper inclusion.

Note that a maximal torus is an abelian subgroup of G. It can be shown that it is a maximal abelian subgroup of G, but not all maximal abelian subgroups of G are torii.

A maximal torus T comes with the action of a group on it, the Weyl group:

Definition 2 (Weyl Group). *Given a maximal torus T in a connected, compact Lie group G, the normalizer of T is the subgroup*

$$N(T) = \{g \in G : gT = Tg\} = \{g \in G : gTg^{-1} = T\}$$

T is a normal subgroup of $N(T)$ and the quotient group

$$W(G,T) = N(T)/T$$

is called the Weyl group of G.

We will later see that all choices of T are conjugate, so different T lead to isomorphic $W(G,T)$. $W(G/T)$ acts on T since $N(T)$ acts by conjugation

$$(n,t) \in N \times T \to ntn^{-1} \in T$$

and its subgroup T acts trivially.

As well as considering the normalizer $N(T)$ of the entire torus T, for each element of $t \in T$, one can consider the normalizer $N(t)$ of that element. This will be of different dimension for different elements.

Definition 3 (Regular and Singular Elements). *For each $t \in T$, define*

$$N(t) = \{g \in G : gtg^{-1} = t\}$$

If dim $N(t) =$ dim T, t is said to be a regular element of T.
If dim $N(t) >$ dim T, t is said to be a singular element of T

For the simple example $G = SU(2)$, one can choose T to be the $U(1)$ subgroup of matrices of the form

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

The space G/T in this case is $S^2 = \mathbf{CP}^1$. The map

$$SU(2) = S^3 \rightarrow SU(2)/U(1) = S^2$$

is the Hopf fibration.

The normalizer $N(T)$ of T consists of two distinct classes of matrices:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$$

So the Weyl group of $SU(2)$ is the group of two elements, with representatives in $N(T)$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The non-trivial element of the Weyl group acts by interchanging $e^{i\theta}$ and $e^{-i\theta}$. Elements of T such that $\theta \neq 0, \pi$ are regular elements, $\theta = 0$ and $\theta = \pi$ are the two singular elements since their normalizer is the entire group.

For a more general class of examples, consider the group $G = U(n)$ of unitary n by n matrices. In this case the maximal torus T is the subgroup of all diagonal matrices

$$T = diag(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n})$$

and the Weyl group is the symmetric group S_n, acting on T by permuting the θ_i. Regular elements of T are those for which the θ_i are all distinct, singular elements are those for which at least two of the θ_i are equal.

It is a theorem in linear algebra that given any unitary matrix M, one can diagonalize it by finding another unitary matrix g such that gMg^{-1} is diagonal with entries given by the eigenvalues of M. The choices of how one diagonalizes leads to different permutations of the eigenvalues along the diagonal.

The fact that any unitary matrix can be diagonalized has a generalization valid for any compact connected Lie group

Theorem 1. *If G is a compact connected Lie group and T is a maximal torus of G, then any element of G is conjugate to an element of T.*

Proof: The simplest proof of this theorem uses a basic result from topology. For a proof not using topology, see [2], section VIII.1. For a more detailed version of the topological proof, see [1] Theorem 4.21. One can reformulate the problem of finding an element $x \in G$ such that

$$x^{-1}gx \in T$$

i.e.

$$x^{-1}gx = t \text{ so } gx = xt$$

which implies

$$gxT = xT$$

the statement that the action of g on the space G/T has a fixed point.

For each element g, one can think of it as a map

$$f_g : yT \in G/T \rightarrow gyT \in G/T$$

This map depends continuously on g and since G is connected, g may be continuously deformed to the identity element. This implies that f_g is homotopic to the identity map. Given any continuous self-map f of a space X to itself, there is an induced map f^* on the cohomology groups of X. One can consider the Lefschetz number

$$L(f) = \sum_i (-1)^i Tr(f^* H^i(X))$$

of the map, which is a homotopy invariant. If f can be continuously deformed to the identity

$$L(f) = L(Id) = \sum_i (-1)^i dim(H^i(X)) = \chi(X)$$

The Lefschetz fixed point formula (for the case of isolated fixed points) identifies the Lefschetz number of a map with the sum of the degrees of the induced maps on spheres surrounding the fixed points. In particular, if there are no fixed points, the Lefschetz number and thus the Euler characteristic must be zero. So if $\chi(G/T)$ is non-zero, there must be at least one fixed point of the map f_g, and the theorem is proved.

It turns out that all the cohomology of G/T is in even dimensions and the space is built out of even-dimensional cells. As a result, the Euler characteristic, which is the sum of the dimensions of the even cohomology groups minus the dimensions of the odd cohomology groups, must be a non-negative number. It turns out that $\chi(G/T)$ is $|W(G,T)|$, the number of elements of the Weyl group. Later on in this course I hope to discuss the computation of the cohomology of G/T. This can be done using Morse theory. The Euler characteristic can also be computed using the formula for the Lefschetz number in terms of local behavior at the fixed points, for this computation, see [1], page 92.

There are various important corollaries of this theorem.

Corollary 1. *Any two maximal torii T_1, T_2 of G are conjugate*

Proof: This can be proved by noting that a torus T always has a generating element, i.e. an element whose powers are dense in T. If θ_i are the angular variables labelling points in the torus, a generating element can be constructed by taking exponentiating a linear combination of these variables, with coefficients independent over $\mathbf{Q}$. If t_2 is a generator of T_2, then by the theorem

$$x^{-1}t_2x = t$$

for some $x \in G$ and $t \in T$. Thus $t_2 \in xTx^{-1}$ and so are all its powers. Since t_2 is a generator, this means that $T_2 \subset xTx^{-1}$. But since T_2 is a maximal torus, we must have $T_2 = xTx^{-1}$

Two other important corollaries:

Corollary 2. *All maximal tori have the same dimension. We will call this common dimension the rank of G*

Corollary 3. *For a compact connected Lie group, the exponential map is surjective.*

This is true since it is clearly true for elements on a maximal torus, and the theorem implies that every element of G is on a maximal torus.

References

[1] Adams, J. Frank, *Lectures on Lie Groups,* University of Chicago Press, 1969.

[2] Simon, B., *Representations of Finite and Compact Groups,*American Mathematical Society, 1996.