Topics in Representation Theory: The Adjoint Representation

1 The Adjoint Representation

Besides the left and right actions of G on itself, there is the conjugation action

$$c(g) : h \to ghg^{-1}$$

Unlike the left and right actions which are transitive, this action has fixed points, including the identity.

Definition 1 (Adjoint Representation). *The differential of the conjugation action, evaluated at the identity, is called the adjoint action*

$$Ad(g) = c_*(g)(e) : T_eG \to T_eG$$

Identifying $\mathfrak{g}$ with T_eG and invoking the chain rule to show that

$$Ad(g_1) \circ Ad(g_2) = Ad(g_1g_2)$$

this gives a homomorphism

$$Ad(g) : G \to GL(\mathfrak{g})$$

called the adjoint representation.

So, for any Lie group, we have a distinguished representation with dimension of the group, given by linear transformations on the Lie algebra. Later we will see that there is an inner product on the Lie algebra with respect to which these transformations are orthogonal.

For the matrix group case, the adjoint representation is just the conjugation action on matrices

$$Ad(g)(y) = gYg^{-1}$$

since one can think of the Lie algebra in terms of matrices infinitesimally close to the unit matrix and carry over the conjugation action to them.

Given any Lie group representation

$$\pi : G \to GL(V)$$

taking the differential gives a representation

$$d\pi : \mathfrak{g} \to End(V)$$

defined by

$$d\pi(X)v = \frac{d}{dt}(\pi(exp(tX))v)|_{t=0}$$

for $v \in V$. Using our previous formula for the derivative of the differential of the exponential map, we find for the adjoint representation $Ad(g)$ that the associated Lie algebra representation is given by

$$ad(X)(Y) = \frac{d}{dt}(c(exp(tX))_*(Y))|_{t=0} = \frac{d}{dt}(Ad(exp(tX))(Y))|_{t=0} = [X, Y]$$

For the special case of matrix groups we can check this easily since expanding the matrix exponential gives

$$e^{tX} Y e^{-tX} = Y + t[X, Y] + O(t^2)$$

So associated to $Ad(G)$, the adjoint representation of the Lie group G on $\mathfrak{g}$, taking the derivative we have $ad(\mathfrak{g})$, a Lie algebra representation of $\mathfrak{g}$ on itself

$$ad(\mathfrak{g}) : X \in \mathfrak{g} \to ad(X) = [X, \cdot] \in End(\mathfrak{g})$$

An important property of the adjoint representation is that there is an invariant bilinear form on $\mathfrak{g}$. This is called the "Killing form", after the mathematician Wilhelm Killing (1847-1823). Killing was responsible for many important ideas in the theory of Lie algebras and their representations, but not for the Killing form. Borel seems to have been the first to use this terminology, but now says he can't remember what inspired him to use it[1].

Definition 2 (Killing Form). *The Killing form on $\mathfrak{g}$ is the bilinear form*

$$K(X, Y) = Tr(ad(X) \circ ad(Y))$$

Here $X \in \mathfrak{g}, Y \in \mathfrak{g}$ and Tr is the trace.

The Killing form has the following important properties:

Theorem 1. *(i) It is symmetric*

$$(X, Y) = K(Y, X)$$

(ii) It is invariant under the adjoint action, i.e. for all $g \in G$

$$K(Ad(g)X, Ad(g)Y) = (X, Y)$$

(iii) For each $Z \in \mathfrak{g}$, the endomorphism $ad(Z)$ is skew-symmetric with respect to the Killing form, i.e.

$$K(ad(Z)X, Y) = -K(X, ad(Z)Y)$$

Proof (i) This follows from the symmetry of the trace

$$Tr(AB) = Tr(BA)$$

(ii) We have

$$ad(Ad(g)X) = Ad(g)ad(X)(Ad(g))^{-1}$$

so

$$\begin{aligned}
K(Ad(g)X, Ad(g)Y) &= Tr(Ad(g)ad(X)(Ad(g))^{-1} \circ Ad(g)ad(Y)(Ad(g))^{-1}) \\
&= Tr(Ad(g)ad(X)ad(Y)(Ad(g))^{-1}) \\
&= Tr((Ad(g))^{-1}Ad(g)ad(X)ad(Y)) \\
&= Tr(ad(X)ad(Y)) \\
&= K(X, Y)
\end{aligned}$$

by cyclicity of the trace.

(iii)If we take

$$g = exp(tZ)$$

then differentiating (ii) at $t = 0$ gives

$$K(ad(Z)X, Y) + K(X, ad(Z)Y) = 0$$

We will see later that for semi-simple compact Lie groups, the Killing form is non-degenerate and its negative is a positive definite inner product on $\mathfrak{g}$.

As usual, the simplest example to keep in mind is $G = SU(2)$. In this case the Lie algebra $\mathfrak{su}(2)$ has a basis of skew-hermitian 2 by 2 matrices, these span the tangent space $\mathbf{R}^3$ to the group at the identity, which is that tangent space to S^3. The adjoint group action on this $\mathbf{R}^3$ is an action by orthogonal tranformations in $SO(3)$. The Killing form is just the negative of the standard inner product on

Using Pauli matrices, a standard basis is:

$$S_1 = -i\sigma_1/2 = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix}, \quad S_2 = -i\sigma_2/2 = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix},$$

$$S_3 = -i\sigma_3/2 = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}$$

and these satisfy

$$[S_j, S_k] = \epsilon_{jkl} S_l$$

where ϵ_{jkl} is a symbol antisymmetric in its indices and such that it is 1 for ϵ_{123} and all cyclic permutations of the indices (123).

Writing an element $X \in \mathfrak{su}(2)$ as

$$X = x_1 S_1 + x_2 S_2 + x_3 S_3$$

the adjoint group action on X by an element $g \in SU(2)$ is the map

$$X \to gXg^{-1}$$

and this takes the vector

$$\mathbf{x} = (x_1, x_2, x_3)$$

to a new vector

$$\mathbf{x}' = (x_1', x_2', x_3')$$

where

$$\mathbf{x}' = A\mathbf{x}$$

for some matrix $A \in SO(3)$.

The adjoint action of the Lie algebra on itself is given by the commutation relations for S_k

$$ad(S_k) : X \to [S_k, X]$$

One can work out what this means explicitly in terms of matrices, for instance

$$ad(S_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

References

[1] Borel, A., *Essays in the History of Lie Groups and Algebraic Groups,*American Mathematical Society, 2001, Page 5.