TOPICS IN REPRESENTATION THEORY: HAMILTONIAN MECHANICS AND SYMPLECTIC GEOMETRY

We'll now turn from the study of specific representations to an attempt to give a general method for constructing Lie group representations. The idea in question sometimes is called "geometric quantization." Starting from a classical mechanical system with symmetry group G, the corresponding quantum mechanical system will have a Hilbert space carrying a unitary representation of G and the hope is that many if not most irreducible representations can be constructed in this way. The first step in such a program involves understanding what sort of mathematical structure is involved in a classical mechanical system with a Lie group G of symmetries. This material is fairly standard and explained in many places, two references with many more details are [1] and [2].

1 Hamiltonian Mechanics and Symplectic Geometry

The standard example of classical mechanics in its Hamiltonian form deals with a single particle moving in space (\mathbf{R}^3). The state of the system at a given time tis determined by six numbers, the coordinates of the position (q_1, q_2, q_3) and the momentum (p_1, p_2, p_3) . The space \mathbf{R}^6 of positions and momenta is called "phase space." The time evolution of the system is determined by a single function of these six variables called the Hamiltonian and denoted H. For the case of a particle of mass m moving in a potential $V(q_1, q_2, q_3)$,

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(q_1, q_2, q_3)$$

The time evolution of the state of the system is given by the solution of the following equations, known as Hamilton's equations

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$
$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

and there is an obvious generalization of this to a phase space \mathbf{R}^{2n} of any even dimension.

A more obvious set of similar equations is the equations for a gradient flow in 2n dimensions

$$\frac{dp_i}{dt} = -\frac{\partial f}{\partial p_i}$$
$$\frac{dq_i}{dt} = -\frac{\partial f}{\partial q_i}$$

These equations correspond to flow along a vector field ∇_f which comes from choosing a function f, taking -df, then using an inner product on \mathbf{R}^{2n} to dualize and get a vector field from this 1-form. In other words we use a symmetric non-degenerate 2-form (the inner product $\langle \cdot, \cdot \rangle$) to produce a map from functions to vector fields as follows:

$$f \to \nabla_f : \langle \nabla_f, \cdot \rangle = -df$$

Hamilton's equations correspond to a similar construction, with the symmetric 2-form coming from the inner product replaced by the antisymmetric 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

In this case, starting with a Hamiltonian function H, one produces a vector field X_H as follows

$$H \to X_H: \quad \omega(X_H, \cdot) = i_{X_H}\omega = -dH$$

Hamilton's equations are then the dynamical system for the vector field X_H . Here ω is called a symplectic form and X_H is sometimes called the symplectic gradient of H. While the flow along a gradient vector field of f changes the value of f as fast as possible, flow along X_H keeps the value of H constant since

$$dH = -\omega(X_H, \cdot)$$
$$dH(X_H) = -\omega(X_H, X_H) = 0$$

since ω is antisymmetric.

One can check that the equation

$$i_{X_H}\omega = -dH$$

implies Hamilton's equations for X_H since equating

$$-dH = -\sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i$$

and

$$i_{X_H} \sum_{i=1}^n dp_i \wedge dq_i$$

implies

$$X_H = -\sum_{i=1}^n \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} + \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i}$$

Another important property of X_H is that

$$\mathcal{L}_{X_H}\omega = (di_{X_H} + i_{X_H}d)\omega = d(-dH) = 0$$

since $d\omega = 0$ (where \mathcal{L}_{X_H} is the Lie derivative with respect to X_H . In general

Definition 1 (Hamiltonian Vector Field). A vector field X that satisfies

$$\mathcal{L}_X \omega = 0$$

is called a Hamiltonian vector field and the space of such vector fields on \mathbf{R}^{2n} will be denoted $Vect(\mathbf{R}^{2n}, \omega)$.

Since ω is non-degenerate, the equation

$$i_{X_f}\omega = -df$$

implies that if $X_f = 0$, then df = 0 and f = constant. As a result, we have an exact sequence of maps

$$0 \to \mathbf{R} \to C^{\infty}(\mathbf{R}^{2n}) \to Vect(\mathbf{R}^{2n},\omega)$$

One can also ask whether all Hamiltonian vector fields (elements of $Vect(\mathbf{R}^{2n}, \omega)$) actually come from a Hamiltonian function. The equation

$$\mathcal{L}_X \omega = (di_X + i_X d)\omega = 0$$

implies

$$di_X\omega = 0$$

so $i_X \omega$ is a closed 1-form. Since $H^1(\mathbf{R}^{2n}, \mathbf{R}) = 0$, this must also be exact, so one can find a Hamiltonian function f.

Just as we saw that df = 0 along X_f , one can compute the derivative of an arbitrary function g along X_f as

$$dg(\cdot) = -\omega(X_g, \cdot)$$
$$dg(X_f) = -\omega(X_g, X_f) = \omega(X_f, X_g)$$

which leads to the following definition

Definition 2 (Poisson Bracket). The Poisson bracket of two functions on \mathbf{R}^{2n}, ω is

$$\{f,g\} = \omega(X_f, X_g)$$

The Poisson bracket satisfies

$$\{f,g\} = -\{g,f\}$$

and

$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0$$

where the second of these equations can be proved by calculating

$$d\omega(X_{f_1}, X_{f_2}, X_{f_3}) = 0$$

These relations show that the Poisson bracket makes $C^{\infty}(\mathbf{R}^{2n})$ into a Lie algebra. As an exercise, one can show that

$$[X_f, X_g] = X_{\{f,g\}}$$

which is the condition that ensures that the map

$$f \in C^{\infty}(\mathbf{R}^{2n}) \to X_f \in Vect(\mathbf{R}^{2n}, \omega)$$

is a Lie algebra homomorphism, with the Lie bracket of vector fields the product in $Vect(\mathbf{R}^{2n}, \omega)$.

So far we have been considering a classical mechanical system with phase space \mathbf{R}^{2n} . The same structures can be defined on an arbitrary manifold satisfying the following definition:

Definition 3 (Symplectic Manifold). A symplectic manifold M is a 2ndimensional manifold with a two-form ω satisfying

- ω is non-degenerate, i.e. for each $m \in M$, the identification of T_m and T_m^* given by ω is an isomorphism
- ω is closed, i.e. $d\omega = 0$.

The two main classes of examples of symplectic manifolds are

• Cotangent bundles: $M = T^*N$.

In this case there is a canonical one-form θ defined at a point $(n, \alpha) \in T^*N$ $(n \in N, \ \alpha \in T^*_n(N))$ by

$$\theta_{n,\alpha}(v) = \alpha(\pi_* v)$$

where π is the projection from T^*N to N. The symplectic two-form on T^*N is

$$\omega = d\theta$$

Physically this case corresponds to a particle moving on an arbitrary manifold M. For the special case $N = \mathbf{R}^n$,

$$\theta = \sum_{i=1}^{n} p_i dq_i$$

• Kähler manifolds. Special cases here include the flag manifolds $G_{\mathbf{C}}/P$ used in the Borel-Weil construction of irreducible representation of G.

On a symplectic manifold M, the same arguments as in \mathbb{R}^{2n} , ω go through and we have an exact sequence of Lie algebra homomorphisms

$$0 \to \mathbf{R} \to C^{\infty}(M) \to Vect(M,\omega) \to H^1(M,\mathbf{R}) \to 0$$

In what follows we will generally be assuming for simplicity that $H^1(M, \mathbf{R}) = 0$.

References

- Bryant, R., An Introduction to Lie Groups and Symplectic Geometry, in Geometry and Quantum Field Theory, Freed, D., and Uhlenbeck, K., eds., American Mathematical Society, 1995.
- [2] Guillemin, V. and Sternberg, S., Symplectic Techniques in Physics, Cambridge University Press, 1984.