Topics in Representation Theory: The Heisenberg Algebra

We'll now turn to a topic which is a precise analog of the previous discussion of the Clifford algebra and spinor representations. By replacing the symmetric two-form (the inner product) in the earlier discussion by an antisymmetric two-form, we get a new algebra, the Heisenberg algebra. The group of automorphism of this algebra is now a symplectic group, and we again get a projective representation of this group, called the metaplectic representation. A similar discussion to ours of these topics can be found in [2] Chapter 17, a much more detailed one in [1].

1 The Heisenberg Algebra and Heisenberg Group

In classical mechanics the state of a particle at a given time t is determined by its position vector $\mathbf{q} \in \mathbf{R}^3$ and its momentum vector $\mathbf{p} \in \mathbf{R}^3$. Heisenberg's crucial idea that lead to quantum mechanics was to take the components of these vectors to be operators on a Hilbert space $\mathcal{H}$, satisfying the commutation relations

$$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0, \quad [P_i, Q_j] = -i\hbar\delta_{i,j}$$

for $i, j = 1, 2, 3$.

The constant $\hbar$ depends on ones choice of units, we'll choose ours so that it is set equal to 1. In quantum mechanics states of a particle at a given time will be vectors in $\mathcal{H}$. A state which is an eigenvector of the operator p_i will be one with a well-defined valued of the corresponding component of the momentum. At this point we won't be doing much real quantum mechanics, just studying the mathematical structure given by a Hilbert space with an action of $2n$ operators satisfying these relations.

An algebra generated by $2n$ elements $\{P_1, \cdots, P_n, Q_1, \cdots, Q_n\}$ satisfying the relations above will be called a Heisenberg algebra and denoted $\mathfrak{h}(n)$. A more invariant formulation of these relations uses the antisymmetric form on $\mathbf{R}^{2n}$ defined by

$$S = \sum_{i=1}^{n} dp_i \wedge dq_i$$

where p_i, q_i are coordinates on $\mathbf{R}^{2n}$. More explicitly this is the antisymmetric form such that

$$S((\mathbf{p}, \mathbf{q}), (\mathbf{p}', \mathbf{q}')) = \mathbf{p} \cdot \mathbf{q}' - \mathbf{q} \cdot \mathbf{p}'$$

One can write the commutation relations using S as

$$[\sum_i (p_i P_i + q_i Q_i), \sum_j (p'_j P_j + q'_j Q_j)] = -iS((\mathbf{p}, \mathbf{q}), (\mathbf{p}', \mathbf{q}'))$$

Note that this is very much like the relations that define the Clifford algebra. In that case vectors v, w corresponded to elements of the algebra and satisfied the relation

$$\{v, w\} = -2Q(v, w)$$

here also $v, w \in \mathbf{R}^{2n}$ correspond to elements of the algebra and satisfy

$$[v, w] = -iS(v, w)$$

anticommutators have been replaced by commutators and the symmetric 2-form Q is replaced by the antisymmetric 2-form S. One could define the Heisenberg algebra as the quotient of the tensor algebra $T(\mathbf{R}^{2n})$ by the ideal I generated by elements of the form $v \otimes w + iS(v, w)$.

It is better to think of the Heisenberg commutation relations as the defining relations for a $2n+1$ dimensional Lie algebra, so we'll use the following definition:

Definition 1 (Heisenberg Lie Algebra). *The Heisenberg Lie algebra $\mathfrak{h}_n$ is the $2n + 1$ dimensional real Lie algebra with basis elements*

$$\{P_1, \cdots, P_n, Q_1, \cdots, Q_n, C\}$$

and Lie bracket defined by

$$[P_i, P_j] = [Q_i, Q_j] = [P_i, C] = [Q_i, C] = [C, C] = 0, \quad [P_i, Q_j] = C\delta_{ij}$$

Here C is a basis vector in the Lie algebra, the relation to our previous commutation relations is that they correspond to the case of C acting by $-i$. The Lie algebra $\mathfrak{h}_n$ is nearly commutative. It is an extension of the commutative Lie algebra $\mathbf{R}^{2n}$ by $\mathbf{R}$, i.e. we have a sequence of homomorphisms

$$0 \to \mathbf{R} \to \mathfrak{h}_n \to \mathbf{R}^{2n} \to 0$$

The $\mathbf{R}$ is in the center of $\mathfrak{h}_n$, so this is a central extension.

$\mathfrak{h}_n$ is isomorphic to a Lie algebra of upper triangular matrices. For example if $n = 1$, $pP + qQ + cC$ can be identified with

$$\begin{pmatrix} 0 & p & c \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}$$

and the Lie bracket is just the matrix commutator since

$$[\begin{pmatrix} 0 & p & c \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & p' & c' \\ 0 & 0 & q' \\ 0 & 0 & 0 \end{pmatrix}] = \begin{pmatrix} 0 & 0 & pq' - pq' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One aspect of the Heisenberg algebra story that does not have an analog in the Clifford algebra case is that since $\mathfrak{h}_n$ is a Lie algebra, one can exponentiate and get a Lie group. This Lie group is generally called the Heisenberg group by mathematicians and denoted $\mathbf{H}_n$. Heisenberg never considered this group since for most purposes in physics just the Lie algebra relations are needed. It was first defined by Weyl and physicists often refer to it as the "Weyl group", but that name is already taken among mathematicians as we have seen.

We can exponentiate using the Baker-Campbell-Hausdorff formula (which simplifies since all commutators of order higher than two are zero)

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$$

so the group law for $\mathbf{H}_n$ can be written

$$(\mathbf{p},\mathbf{q},c)\cdot(\mathbf{p}',\mathbf{q}',c') = (\mathbf{p}+\mathbf{p}',\mathbf{q}+\mathbf{q}',c+c'+\frac{1}{2}S((\mathbf{p},\mathbf{q}),(\mathbf{p}',\mathbf{q}')))$$

2 The Symplectic Group

Since the Heisenberg algebra and group are both defined using the antisymmetric form S, the group of linear transformations of $\mathbf{R}^{2n}$ that leave this form invariant acts as automorphisms in both cases. This group is called the symplectic group $Sp(2n,\mathbf{R})$ and it is the analog in this case of the orthogonal group $SO(2n)$ in the Clifford algebra case. Note that this is *not* the classical compact group $Sp(2n)$, although there is a relation between the two groups in that the complexification $Sp(2n,\mathbf{C})$ satisfies

$$Sp(2n,\mathbf{C})\cap U(2n) = Sp(n)$$

.

The group $Sp(2n,\mathbf{R})$ is non-compact, but we can try and produce a projective representation of it using an analogous construction to the one used to produce the projective spinor representation of $SO(2n)$. This infinite dimensional representation will be called the metaplectic representation and the double cover $Mp(2n,\mathbf{R}) = \widetilde{Sp(2n,\mathbf{R})}$ of $Sp(2n,\mathbf{R})$ analogous to the spin double cover will be called the metaplectic group.

In the Heisenberg case there is no analog of the identification of the Clifford algebra with a matrix algebra and thus the endomorphisms of a vector space of spinors. However, at the group level we have the result

Theorem 1 (Stone-von Neumann Theorem). *Once one fixes a non-zero scalar with which the central element C acts, there is a unique irreducible representation of the Heisenberg group $\mathbf{H}_n$.*

We won't try and give the proof of this here since the analysis is non-trivial due to the fact that the representation is infinite dimensional. For the proof, see [1] chapter 1.5. The importance of this theorem in quantum mechanics is that it tells us that once we have chosen a non-zero Planck's constant, there is a unique irreducible representation of the Heisenberg algebra (at least a unique one that can be exponentiated). This is why physicists are able to use the Heisenberg commutation relations to do calculations, without worry about what they are being represented on.

If

$$U_a : a \in \mathbf{H}_n \to U(\mathcal{H})$$

is the unique irreducible representation by elements of $U(\mathcal{H})$, unitary transformations on a Hilbert space $\mathcal{H}$, and $g \in Sp(2n, \mathbf{R})$ acts via automorphisms

$$a \to g \cdot a$$

on $\mathbf{H}_n$, then $U_{g \cdot a}$ must be unitarily equivalent to U_a, so we can find a unitary operator $R(g)$ such that

$$U_{g \cdot a} = R(g) U_a R(g)^{-1}$$

The operators $R(g)$ are only determined up to scalars and give a projective representation of $Sp(2n, \mathbf{R})$ on $\mathcal{H}$. This will be a true representation of the metaplectic group, the double cover $\widetilde{Sp(2n, \mathbf{R})}$. We will be constructing this more explicitly later on, but note that the metaplectic group is an example of a group which is not a matrix group. While the group $Spin(n)$ is not defined in terms of matrices, it does have a faithful finite-dimensional representation on the spinor vector space of dimension 2^n, so it is a group of 2^n by 2^n complex matrices. In the case of the metaplectic group, there are no finite dimensional representations so one cannot represent it as a group of finite matrices.

3 Canonical Commutation Relations

We will now begin the construction of the metaplectic representation, by analogy with our previous construction of the spinor representation. Just as in that case, we begin by choosing a complex structure J on $\mathbf{R}^{2n}$. In this case we need to choose a J that preserves the symplectic structure, i.e.

$$S(u, v) = S(Ju, Jv)$$

. Later we'll discuss in detail the space of all such choices, but for now, we'll just pick a standard one, identifying the holomorphic coordinates as $w_k = p_k + iq_k$, the anti-holomorphic ones as $\bar{w}_k = p_k - iq_k$.

In the case of the spinor representation, we used an algebra generated by basis elements satisfying the Canonical Anticommutation Relations (CAR). In this case we will use an analogous algebra with anticommutators replaced by commutators, this is called the algebra of Canonical Commutation Relations (CCR). It has $2n$ generators $a_k, a_k^\dagger$ for $k = 1, \cdots, n$ satisfying the relations

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, [a_j, a_k^\dagger] = \delta_{jk}$$

Just as the Clifford algebra relations could be expressed in terms of the CAR, the Heisenberg algebra and the CCR have a similar relation. Setting

$$a_k = \frac{1}{\sqrt{2}}(P_k - iQ_k), \quad a_k^\dagger = \frac{1}{\sqrt{2}}(P_k + iQ_k)$$

relates the generators of the CCR to the generators of the Heisenberg algebra.

The CAR algebra is represented on the exterior algebra $\Lambda^*(\mathbf{C}^n)$, similarly the CCR can be represented on the symmetric algebra $S^*(\mathbf{C}^n)$, or, equivalently, the polynomial ring $\mathbf{C}[w_1, \cdots, w_n]$. Just as in the exterior algebra case our generators corresponded to exterior and interior multiplication by a basis element, here $a_k^\dagger$ acts by multiplication

$$a_k^\dagger \to w_k \cdot$$

and a_k acts by differentiation

$$a_k \to \frac{\partial}{\partial w_k}$$

We will see that one can put a norm on $\mathbf{C}[w_1, \cdots, w_n]$ such that a_k and $a_k^\dagger$ are adjoints. This representation of the CCR and thus the Heisenberg algebra is called the Fock (or Bargmann-Fock) representation.

Physically this representation corresponds to n harmonic oscillators, with the vector $1 \in \mathbf{C}[w_1, \cdots, w_n]$ as the vacuum state and $a_k^\dagger$ the operator that adds one quantum of type k to the vacuum state. This representation is also sometimes known as the oscillator representation. As in the spinor case, the vacuum state depends on the choice of J.

References

[1] Folland, G., *Harmonic Analysis in Phase Space,* Princeton University Press, 1989.

[2] Segal, G., Lie Groups, in *Lectures on Lie Groups and Lie Algebras,* Cambridge University Press, 1995.