Topics in Representation Theory: Clifford Algebras

So far in this course we have given a very general theory of compact Lie groups and their representations, but the only examples we have considered in any detail concern the unitary groups $SU(n)$. Here we have seen that there is a set of $n-1$ fundamental representations and that these can be explicitly realized as the defining representation of $SU(n)$ on $\mathbf{C}^n$ and the corresponding representations on exterior powers $\Lambda^k(\mathbf{C}^n)$ for $k = 1, 2, \cdots n-1$. Other representations of $SU(n)$ are on symmetric powers $S^k(\mathbf{C}^n)$, these are the representations on homogeneous polynomials of n variables. Later on in this course we will study more generally the issue of the representation of $SU(n)$ on general tensors (element of $\mathbf{C}^n \otimes \mathbf{C}^n \otimes \cdots \otimes \mathbf{C}^n$). The geometric picture of Borel-Weil theory relates each of these fundamental representations to a line bundle over a space $SL(n, \mathbf{C})/P$, with $\Lambda^k(\mathbf{C}^n)$ related to a line bundle over $Gr(k, n)$, the Grassmanian of k-planes in $\mathbf{C}^n$. Furthermore, each of these is related to a node on the Dynkin diagram.

The three other infinite classes of classical groups are $Sp(n)$, $SO(2n)$ and $SO(2n+1)$. We will not work out what happens for $Sp(n)$, but will now turn to the case of the orthogonal groups $SO(n)$. For these groups, one again has irreducible fundamental representations on $\Lambda^k(\mathbf{R}^n)$, but there are new representations which cannot be constructed using tensors, called the spinor representations. These are only projective representations of $SO(n)$, but are true representations of the double cover $Spin(n)$. To understand these representations it is convenient to introduce a new algebraic structure called a Clifford algebra, a structure which will include the groups $Spin(n)$, but much else besides.

1 Clifford Algebras

A Clifford algebra is associated to a vector space V with inner product, in much the same way as the exterior algebra $\Lambda^* V$ is associated to V. The multiplication in the Clifford algebra is different, taking into account the inner product. One way of thinking of a Clifford algebra is as $\Lambda^* V$, with a different product, one that satisfies

$$v \cdot v = - < v, v > 1 = -||v||^2 1$$

for $v \in V$. More generally, one can define a Clifford algebra for any vector space V with a quadratic form $Q(\cdot, \cdot)$ (we will be interested in the quadratic form associated to an inner product $Q(\cdot, \cdot) =< \cdot, \cdot >$)

Definition 1. *The Clifford algebra $C(V, Q)$ associated to a real vector space V with quadratic form Q can be defined as*

$$C(V, Q) = T(V)/I(V, Q)$$

where $T(V)$ is the tensor algebra

$$T(V) = \mathbf{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$$

and $I(V,Q)$ is the ideal in $T(V)$ generated by elements

$$v \otimes v + Q(v,v)1$$

(where $v \in V$).

The tensor algebra $T(V)$ is $\mathbf{Z}$-graded, and since $I(V,Q)$ is generated by quadratic elements the quotient $C(V,Q)$ retains only a $\mathbf{Z}_2$ grading

$$C(V,Q) = C_{\mathrm{even}}(V,Q) \oplus C_{\mathrm{odd}}(V,Q)$$

Note the following facts about the Clifford algebra:

- If $Q = 0$, one recovers precisely the definition of the exterior algebra, so

  $$\Lambda^*(V) = C(V, Q = 0)$$

- Applying the defining relation for the Clifford algebra to a sum $v + w$ of two vectors gives

  $$\begin{aligned} (v+w)\cdot(v+w) = v^2 + vw + wv + w^2 &= -Q(v+w, v+w) \\ &= -Q(v^2) - 2Q(v,w) - Q(w^2) \end{aligned}$$

  so the defining relation implies

  $$vw + wv = -2Q(v,w)$$

  which could be used for an alternate definition of the algebra. Also note that in our case where $Q =< \cdot, \cdot >$, this means that two vectors v and w anticommute when they are orthogonal.

- About half of the math community uses the definition given here for the defining relation of a Clifford algebra, the other half uses the relation with the opposite sign

  $$v \cdot v = ||v||^2 1$$

- Non-degenerate quadratic forms over a real vector space of dimension n can be put in by a change of basis into a canonical diagonal form with p $+1$'s and q -1's on the diagonal, $p+q = n$. We will mostly be interested in studying the Clifford algebra for the case of the standard positive definite quadratic form $p = n$, $q = 0$. Physicists are also quite interested in the case $p = 3$, $q = -1$, which corresponds to Minkowski space, four dimensional space-time equipped with this kind of quadratic form.

  Later on we will be considering the case of complex vector spaces. In this case there is only one non-degenerate Q up to isomorphism (all diagonal elements can be chosen to be $+1$).

- For the case of $V = \mathbf{R}^n$ with standard inner product ($p = n$), we will denote the Clifford algebra as $C(n)$. Choosing an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbf{R}^n$, $C(n)$ is the algebra generated by the e_i, with relations
$$e_i e_j + e_j e_i = -2\delta_{ij}$$

Clifford algebras are well-known to physicists as "gamma matrices" and were intoduced by Dirac in 1928 when he discovered what is known as the "Dirac" equation. Dirac was looking for a version of the Schrödinger equation of quantum mechanics that would agree with the principles of special relativity. One common guess for this was what is now known as the Klein-Gordon equation (units with the speed of light $c = 1$ are being used)

$$\Box\psi = (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2})\psi = 0$$

but the second-order nature of this PDE was problematic, so Dirac was looking for a first-order operator $\not{D}$ satisfying

$$\not{D}^2 = -\Box$$

Dirac found that if he defined

$$\not{D} = \sum_{i=1}^{4} \gamma_i \frac{\partial}{\partial x_i}$$

then this would work if the γ_i satisfied the relations for generators of a Clifford algebra on four dimensional Minkowski space. The Dirac operator remains of fundamental importance in physics, and over the last few decades its importance in mathematics in has become widely realized. For any space with a metric one can define a Dirac operator, which plays the role of a "square-root" of the Laplacian.

One can easily see that, as a vector space $C(n)$ is isomorphic to $\Lambda^*(\mathbf{R}^n)$. Any element of $C(n)$ is a linear combination of finite strings of the form

$$e_{i_1} e_{i_2} \cdots$$

and using the relations
$$e_i e_j = -e_j e_i$$
these can be put into a form where

$$i_1 < i_2 < \cdots$$

eliminating any repeated indices along the way with the relation $e_i^2 = -1$. So, just as for the exterior algebra, the 2^n elements

$$1$$
$$e_i$$
$$e_ie_j \quad i<j$$
$$\cdots$$
$$e_1e_2\cdots e_n$$

form a basis.

More abstractly, the Clifford algebra is a filtered algebra

$$F_0 \subset F_1 \subset \cdots \subset F_n = C(n)$$

with F_i the part of $C(n)$ one gets from multiplying at most n generators. The associated graded algebra to the filtration is the exterior algebra

$$\begin{aligned} \mathrm{gr}_F C(n) &= F_1/F_0 \oplus F_2/F_1 \oplus \cdots \oplus F_n/F_{n-1} \\ &= \Lambda^*(\mathbf{R}^n) \end{aligned}$$

1.1 Examples

Let's now consider what these algebras $C(n)$ actually are for some small values of n.

For $n=1$, $C(1)$ is the algebra of dimension 2 over $\mathbf{R}$ generated by elements $\{1, e_1\}$ with relation $e_1^2 = -1$. This is just the complex numbers $\mathbf{C}$, so $C(1) = \mathbf{C}$.

For $n=2$, $C(2)$ is the algebra of dimension 4 over $\mathbf{R}$ generated by elements $\{1, e_1, e_2\}$ with relations

$$e_1^2 = -1,\ e_2^2 = -1,\ e_1e_2 = -e_2e_1$$

This turns out to be precisely the quaternion algebra $\mathbf{H}$ under the identification

$$i = e_1,\ j = e_2,\ k = e_1e_2$$

so $C(2) = \mathbf{H}$.

For higher values of n and for arbitray signature of the quadratic form, see chapter 1 of [1] for a calculation of what all these real Clifford algebras are. We'll just quote the result here:

$$C(3) = \mathbf{H} \oplus \mathbf{H},\ C(4) = M(2, \mathbf{H}),\ C(5) = M(4, \mathbf{C})$$

$$C(6) = M(8, \mathbf{R}),\ C(7) = M(8, \mathbf{R}) \oplus M(8, \mathbf{R}),\ C(8) = M(16, \mathbf{R})$$

and for higher values of n, things are periodic with period 8 since

$$C(n+8) = C(n) \otimes M(16, \mathbf{R})$$

References

[1] Lawson, B. and Michelson, M., *Spin Geometry,* Princeton University Press, 1989.