Topics in Representation Theory: Fundamental Representations and Highest Weight Theory

We'll now start the study of arbitrary irreducible representations of higher rank compact Lie groups, beginning with the purely Lie-algebraic aspects of the story. A fundamental weakness is the Lie-algebraic approach is the lack of any analog of the explicit representation in terms of homogeneous polynomials that we were able to use in the $SU(2)$ case. Working just with the Lie algebra and its commutation relations, one can derive many properties of irreducible representations, but the only actual construction of the representations in this context are rather inexplicit, involving taking quotients of infinite dimensional modules of the enveloping algebra known as Verma modules. We will not construct representations this way, instead using the Borel-Weil construction, a geometric technique that works at the level of the group and generalizes the homogeneous polynomial construction of the $SU(2)$ case. For now we will describe some of the Lie-algebraic techniques that are both computationally useful and allow the precise formulation of the general picture, postponing proofs that these actually give all irreducible representations until we use geometric techniques to construct them.

In this lecture we'll be following [1] chapter 14 fairly closely. As a result these notes may be even sketchier than usual.

1 Co-roots and Fundamental Weights

For each simple root α, we would like to identify a copy of $\mathfrak{sl}(2,\mathbf{C}) \subset \mathfrak{g}_{\mathbf{C}}$, which we will call $\mathfrak{sl}(2,\mathbf{C})_\alpha$. The weights of our representations will then be classified by how they behave under the maximal abelian subalgebras of each of these $\mathfrak{sl}(2,\mathbf{C})_\alpha$. Recall that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$$

and

$$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{t}_{\mathbf{C}}$$

We can choose a set of three generators $X_\alpha^+, X_\alpha^-, H_\alpha$ satisfying the $\mathfrak{sl}(2,\mathbf{C})$ commutation relations, with $X_\alpha^+ \subset \mathfrak{g}_\alpha$, $X_\alpha^- \subset \mathfrak{g}_{-\alpha}$ and $H_\alpha \subset \mathfrak{t}_{\mathbf{C}}$

$$[H_\alpha, X_\alpha^+] = \alpha(H_\alpha)X_\alpha^+ = 2X_\alpha^+$$

$$[H_\alpha, X_\alpha^-] = -\alpha(H_\alpha)X_\alpha^- = -2X_\alpha^-$$

$$[X_\alpha^+, X_\alpha^-] = H_\alpha$$

We can also choose X_α^- to be the conjugate of X_α^+. The element $H_\alpha \in \mathfrak{t}$ is canonically associated with the root α and is called the co-root of α and sometimes denoted $\alpha^\vee$. It can be defined as follows

Definition 1 (Co-root). *The co-root H_α associated to a root α is the unique element in $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ satisfying $\alpha(H_\alpha) = 2$.*

Our representation of $\mathfrak{g}$ will also be a representation of each of the $\mathfrak{sl}(2, \mathbf{C})_\alpha$. We have seen that representations of $\mathfrak{sl}(2, \mathbf{C})$ decompose into weight spaces V_β with integral eigenvalues of H. For each co-root H_α the weight-spaces V_β of our representation V must satisfy $H_\alpha V_\beta = \beta(\alpha) \in \mathbf{Z}$.

Recall that a lattice Λ in a vector space V is a discrete additive subgroup, closed in V, such that element of Λ span V as a vector space. Motivated by the above requirement on the weights of our representations, we'll define

Definition 2 (Weight Lattice). *The lattice of weights $\Lambda_W \subset \mathfrak{t}^*$ is the set of $\beta \in \mathfrak{t}^*$ such that $\beta(H_\alpha) \in \mathbf{Z}$ for all simple roots α.*

Note that the roots themselves define a basis for a lattice, the root lattice Λ_R and $\Lambda_R \subset \Lambda_W$. For simply connected G all elements of Λ_W are actually weights of representations of G, whereas for non-simply connected groups (for example $G = SO(3)$) only a sub-lattice are actually weights, with the full lattice being weights for the universal covering group. For a full discussion of the relation of these various lattices, see [2] section IX.1.

To better understand the relation of the weight lattice to the roots, note that the Weyl group $W(G, T)$ takes weights to weights, leaving the weight lattice invariant. In particular the reflections s_α corresponding to each simple root which generate W take weights to weights. Recall the formula for a Weyl reflection with respect to a root α

$$s_\alpha(\beta) = \beta - \frac{2 < \alpha, \beta >}{< \alpha, \alpha >}\alpha$$

As an exercise (see [1] section 14.2) show that under the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ using the Killing form, the co-root H_α correspond to

$$\frac{2\alpha}{< \alpha, \alpha >}$$

Using this, one can rewrite the Weyl reflection as

$$s_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha$$

Reflecting one root ($\beta = \alpha_j$) with respect to another (α_i) gives

$$s_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2 < \alpha_j, \alpha_i >}{< \alpha_i, \alpha_i >}\alpha_i$$

but

$$\frac{2 < \alpha_j, \alpha_i >}{< \alpha_i, \alpha_i >} = \alpha_j(H_{\alpha_i}) = A_{ji}$$

where A is the Cartan matrix.

The dual basis to the basis of co-roots is called the basis of fundamental weights:

Definition 3. *The set of fundamental weights of G is a set of rank G elements $\omega_i \in \mathfrak{t}^*$ such that*

$$\omega_i(H_{\alpha_j}) = \delta_{ij}$$

In general a weight β can be written using this basis as

$$\beta = \sum_i \beta(H_{\alpha_i})\omega_i$$

so the simple roots are linear combinations of the fundamental weights, with transformation matrix the Cartan matrix

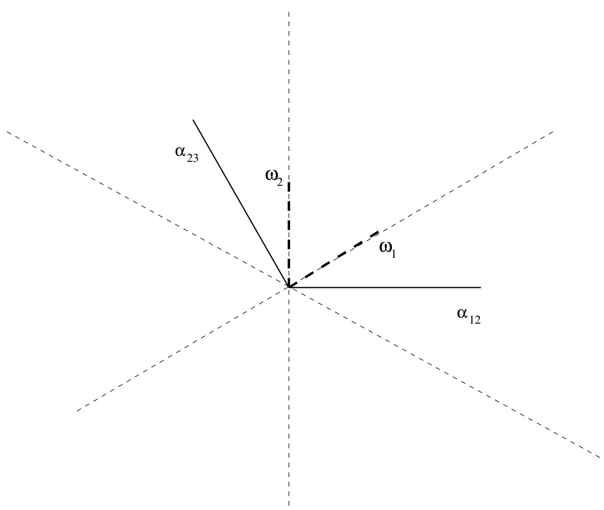
$$\alpha_j = \sum_i \alpha_j(H_{\alpha_i})\omega_i = \sum_i A_{ji}\omega_i$$

As an example consider the case $G = SU(3)$ where, defining $\alpha_1 = \alpha_{12}, \alpha_2 = \alpha_{23}$ and knowing that the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

we have

$$\begin{aligned} \alpha_1 &= 2\omega_1 - \omega_2 \\ \alpha_2 &= -\omega_1 + 2\omega_1 \end{aligned}$$



Simple Roots and Fundamental Weights for $SU(3)$

As a more general example, for $G = SU(n)$, $\mathfrak{t}$ consists of diagonal matrices with zero trace and and entries $D_{ii} = \lambda_i$, and the roots are

$$\alpha_{ij}(\lambda) = \lambda_i - \lambda_j$$

for $i \neq j$. One choice of positive roots are those for which $i < j$, the simple roots can be chosen to be $\alpha_{i,i+1}$. The co-roots are

$$H_{\alpha_{i,i+1}} = E_{ii} - E_{i+1,i+1}$$

where E_{ij} is the matrix that whose i, j'th entry is one, all others zero. The corresponding fundamental weights are

$$\omega_{i,i+1}(\lambda) = \lambda_1 + \cdots + \lambda_i$$

2 Highest Weight Theorem

Now we can at least state the recipe for constructing, first we'll define

Definition 4. *A weight β is dominant if $< \beta, \alpha_i > \geq 0$ for all simple roots α_i, i.e. it is in the closure of the fundamental Weyl chamber.*

and

Definition 5. *For a representation of G on V, $v \in V$ is called a highest weight vector if for all positive roots α, $X_\alpha \in \mathfrak{g}_\alpha$ implies $X_\alpha = 0$. If the highest weight vector v is in the weight space V_β, β is the highest weight of the representation.*

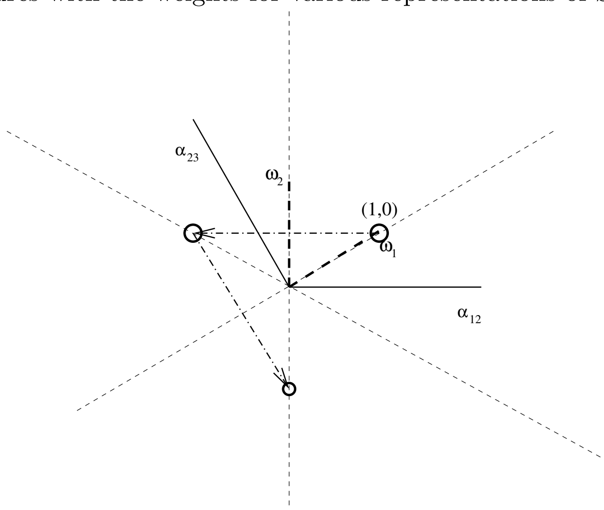
Now the recipe to construct an irreducible representation goes as follows: pick a dominant weight λ. Assume one has a representation with this highest weight and pick a highest weight vector. Now applying all possible combinations of elements of negative root spaces to this vector generates the full representation. This motivates the following theorem

Theorem 1 (Highest Weight Theorem). *For any dominant weight $\lambda \in \Lambda_W$ there exists a unique, irreducible, finite-dimensional representation V_λ of G with highest weight λ.*

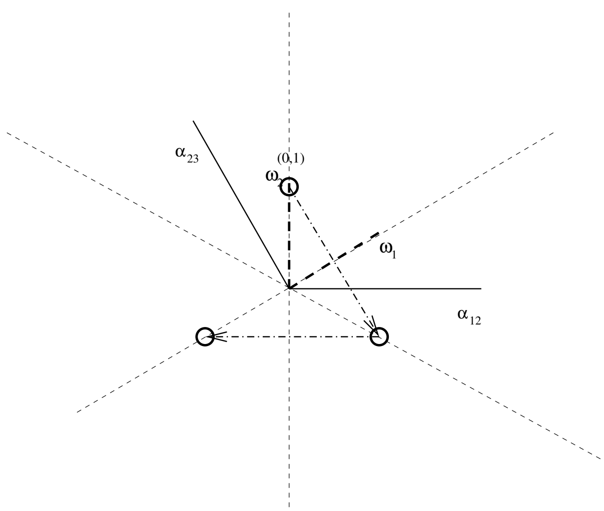
For a "half proof" see [1] 14.18. The tricky part of really proving this theorem involves the existence, since one doesn't have any kind of construction to start with to get a highest weight vector. Later on when we examine the Borel-Weil geometric picture of representations we will have an explicit construction that can be used to prove existence.

The Weyl-invariant geometry of the weight lattice provides a good hold on the pattern that the weights of an irreducible representation must lie in. They lie inside the convex hull of the figure one gets by acting on the highest weight with elements of the Weyl group. To get not just the weights, but the multiplicity with which each weight occurs, one needs to know the character of the representation and we will turn next lecture to that issue.

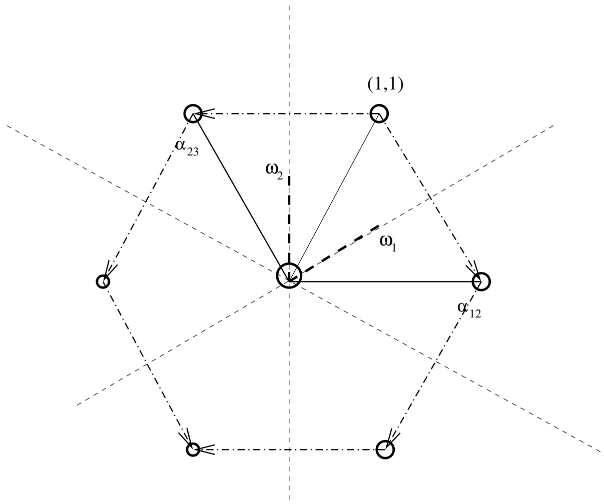
Since the fundamental weights form a basis for the weight lattice and the dominant weights are the ones that are non-negative integral linear combinations of the fundamental weights, all irreducible weights can be characterized by a set of rank G non-negative integers. One can think of each of these integers as being associated to the corresponding node of the Dynkin diagram for the group. Some figures with the weights for various representations of $SU(3)$ follow.



Weights for the $(1,0)$ fundamental representation of $SU(3)$



Weights for the $(0,1)$ fundamental representation of $SU(3)$



Weights for the $(1,1)$ adjoint representation of $SU(3)$

References

[1] Fulton, W., Harris, J., *Representation Theory: A First Course,* Springer-Verlag, 1991.

[2] Simon, B., *Representations of Finite and Compact Groups,* American Mathematical Society, 1996.