Modern Geometry, Fall 2017: Problem Set 4

Due Thursday, October 5

Problem 1: Show that the Lie derivative $\mathcal{L}_X$ is a derivation, i.e.

$$\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$$

for $\alpha, \beta \in \Omega^*(M)$.

Problem 2: Prove the two Cartan formulas relating the Lie derivative L_X, the interior product i_X and the exterior derivative d

a) $\mathcal{L}_X = d i_X + i_X d$

b) $\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}$

Problem 3: Show that, for M a smooth manifold, $X_1, \ldots, X_{k+1}$ smooth vector fields on M, and $\omega \in \Omega^k(M)$

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{k+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \end{aligned}$$

Problem 4: Using the definition of the Poisson bracket of two functions in terms of a symplectic 2-form ω:

$$\{f, g\} = \omega(X_f, X_g)$$

(where X_f is determined by $i_{X_f}\omega = df$)
show that $d\omega = 0$ is equivalent to the Jacobi identity for the Poisson bracket, i.e. for three functions f_1, f_2, f_3 one has

$$\{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\} = 0$$