Modern Geometry, Fall 2017: Problem Set 2

Due Thursday, September 21

Problem 1: Show that if $F : M_1 \to M_2$ and $G : M_2 \to M_3$ are smooth maps between smooth manifolds, then the differentials satisfy

$$(G \circ F)_* = G_* \circ F_*$$

Show that for $M_1 = \mathbf{R}$, $M_2 = \mathbf{R}^n$ and $M_3 = \mathbf{R}$ this is the usual chain rule of multivariable calculus

Problem 2: If f and g are smooth functions on a smooth manifold M, and X and Y are smooth vector fields on M, show that the Lie bracket satisfies

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$$

Problem 3: Starting from the definition of smooth differential 1-forms as smooth sections $M \to T^*M$, find an appropriate definition of a smooth structure on T^*M and show that such smooth differential 1-forms ω when expressed in local coordinates as

$$\omega = \sum_i a_i dx^i$$

are given by smooth coefficient functions a_i.

Problem 4:

The "canonical 1-form" θ on a cotangent bundle T^*M is defined at the point $(p, l) \in T^*M$ ($p \in M$, and $l \in (T_pM)^*$) by

$$\theta_{(p,l)}(X) = l(\pi_*(X)$$

where X is a tangent vector to T^*M at (p, l).
For $M = \mathbf{R}^n$ find an expression for θ in terms of coordinates.