Modern Geometry, Fall 2017: Problem Set 10

Due Thursday, November 30

Problem 1: Show that the fundamental equations for the curvature and torsion of a $GL(n, \mathbf{R})$ frame bundle can be written as

$$\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j$$

$$\Theta^i = d\theta^i + \sum_j \omega^i_j \wedge \theta^j$$

where $\Omega^i_j, \Theta^i \in \Omega^2(P)$ are components with respect to a basis of the Lie algebra of $GL(n, \mathbf{R})$ and $\mathbf{R}^n$ respectively.

Problem 2: Starting with any $O(n)$ connection ω' on a frame bundle, show that one can write

$$\Theta^i = \frac{1}{2} \sum_{j,k} T^i_{jk} (\theta^j \wedge \theta^k)$$

for some functions satisfying $T^I_{jk} = -T^i_{kj}$. Then let

$$(\omega'')^i_j = \frac{1}{2} \sum_k (T^i_{jk} + T^j_{ki} + T^k_{ji}) \theta^k$$

and show that

$$\omega = \omega' + \omega''$$

has torsion zero.

Problem 3: Show that for a $GL(n, \mathbf{R})$ connection, under a change of section given by a transition function $\varphi_{\alpha\beta}$, the curvature (pulled back to the base) changes by

$$F_\alpha = s^*_\alpha(\Omega) \to F_\beta = s^*_\beta(\Omega) = \varphi^{-1}_{\alpha\beta} F_\alpha \varphi_{\alpha\beta}$$

Problem 4: For a principal G-bundle P over M, show that $\alpha \in \Omega^k(P)$ can be written $\alpha = \pi^*(\widetilde{\alpha})$ for some $\widetilde{\alpha} \in \Omega^k(M)$ iff

- α is horizontal ($\alpha(X) = 0$ for X in the vertical subspace of the tangent space to P).
- α is invariant under the right action of G on P, meaning

$$R^*_g(\alpha) = \alpha$$

Such forms on P are called *basic* (i.e. "coming from the base").
In addition, show that for a basic form α one has $D\alpha = d\alpha$. Here $D\alpha$ is given on tangent vectors by

$$D\alpha(X_1, X_2, \cdots, X_k) = d\alpha(h(X_1), h(X_2), \cdots, h(X_k))$$

Problem 5: Explicitly find connection one-forms A_α, A_β on the upper and lower hemispheres of a sphere that come from a connection on a non-trivial $U(1)$ bundle P over S^2 with $c_1(P)[S^2] = 1$.

Problem 6: If ρ is the projection $P \times I$ to P (I is interval $[0, 1]$), and ω_0, ω_1 are connections on P (a principal G bundle over M), show that

$$\widetilde{\omega} = (1 - t)\rho^*\omega_0 + t\rho^*\omega_1$$

is a connection on the bundle $P \times I$ over $M \times I$.