# Modern Geometry II: Problem Set 3

**Due Monday, April 18**

**Problem 1:** Prove the the symplectic form $\omega$ of a compact symplectic manifold can never be an exact form. Use this to show that the only for $n = 2$ does the n-sphere $S^n$ have a symplectic structure.

**Problem 2:** Given a symplectic action of a Lie group $G$ on a symplectic manifold $M$, with a moment map

$$\mu : M \to (Lie\ G)^*$$

define a dual "co-moment map"

$$\mu^* : Lie\ G \to C^\infty(M)$$

and show that it satisfies:
a) $\mu^*(X)$ is a hamiltonian function vector field corresponding to $X$.
b) $\mu^*$ is a Lie algebra homomorphism, where the bracket on $C^\infty(M)$ is the Poisson Bracket.

**Problem 3:** Show that complex projective space $\mathbf{C}P^n$ is a complex manifold by explicitly finding a set of complex coordinate charts and holomorphic transition functions between them.

**Problem 4:**

If $J$ is an almost complex structure on a manifold $M$, define its torsion (or Nijenhuis tensor) to be

$$N(X,Y) = [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y]$$

where $X, Y$ are vector fields on $M$. Show that $N$ is the zero tensor if $J$ is integrable.

**Problem 5:** For $E$ a holomorphic vector bundle, define an operator

$$\overline{\partial}_E : \Omega^{p,q}(M, E) \to \Omega^{p,q+1}(M, E)$$

in terms of a local choice of holomorphic frame, such that for $E$ a trivial bundle $\overline{\partial}_E = \overline{\partial}$. Show that your definition is independent of the choice of holomorphic frame, and that it satisfies $\overline{\partial}_E^2 = 0$.