Modern Geometry I: Problem Set 5

Due Monday, December 13

Problem 1: Given a connection 1-form ω on a principal bundle P and the corresponding curvature 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

show that $\Omega(X,Y) = 0$ if either X or Y are in the vertical subspace $T^V P$.

Problem 2: Given a connection on a principal bundle P over M, and a curve $\tau = x_t$ in M parametrized by $t \in [0,1]$. Parallel transport along the horizontal lift of τ gives a map

$$\tilde{\tau}_0^1 : \pi^{-1}(x_0) \to \pi^{-1}(x_1)$$

If τ lies in some coordinate patch U, and one picks a trivialization of P over U, $\tilde{\tau}_0^1$ can be identified with an element of G. How does this element of G transform under change of trivialization over U?

Problem 3: For a vector bundle E over M constructed as an associated vector bundle to a principal bundle P, show that

$$\nabla_X(g\phi) = g\nabla_X\phi + (Xg)\phi$$

where X is a smooth vector field on M, $g \in C^\infty(M)$, and $\phi \in \Gamma(E)$.

Problem 4:

Given a vector bundle $E = P \times_G V$ constructed as an associated bundle to a principal G-bundle over M using a representation ρ of G on$V$, show that

$$\Omega^i(M,E) = \Omega^i_{basic}(P,V)$$

Problem 5:

Given a curvature form Ω on a principal bundle P over M, and two vector fields X and Y on M, show that, on sections $\Gamma(E)$ of an associated vector bundle E, one has

$$\rho_*(\Omega)(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X.Y]}$$