Modern Geometry I: Problem Set 4

Due Monday, November 29

Problem 1: For the Lie group $G = SO(3)$, find an explicit basis for the Lie algebra Lie (G) and identify Lie (G) with $\mathbf{R}^3$. Explicitly construct the adjoint representations

$$Ad : SO(3) \to GL(3, \mathbf{R})$$

of the group, and

$$ad : Lie\ SO(3) \to M(3, \mathbf{R})$$

of the Lie algebra.
Express ad in terms of the vector cross-product on $\mathbf{R}^3$.

Problem 2: Consider the group $Aff(\mathbf{R})$ of affine transformations of $\mathbf{R}$. It can be identified with the subgroup of $GL(2, \mathbf{R})$ of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \neq 0$.
a) find the left invariant and right invariant 1-forms on this group.
b) find the left-invariant Maurer-Cartan form on the group and show that it satisfies the Maurer-Cartan equations.
c) find the left and right invariant 2-forms on the group.

Problem 3: Suppose that $P' \to M$ is a principal H bundle and $H \subset G$ is a Lie subgroup. Show that $P' \times_H G \to M$ is naturally a principal G bundle. A reduction of a G bundle $P \to M$ to an H bundle is a pair consisting of an H bundle $P' \to M$ and an isomorphism of G bundles $P' \times_H G \to P$. Show that a principal G bundle reduces to the subgroup $H = \{1\}$ iff the G bundle is trivial.

Problem 4: Prove that the first two definitions of a connection given in class (as a choice of horizontal subspace, as a 1-form) are equivalent.

Problem 5: Given a connection ω on a principal bundle P and two local sections s_1 and s_2 defined on a coordinate patch U, derive the formula relating $s_1^*\omega$ and $s_2^*\omega$.

Problem 6:

Consider the complex line bundles L_n associated to the Hopf bundle (principal $U(1)$ bundle) $S^3 \to \mathbf{CP}^1$, using the representation of $U(1)$ on $\mathbf{C}$ by $e^{in\theta}$. Find the value of n that corresponds to the tautological line bundle over $\mathbf{CP}^1$. Find the value of n that corresponds to the tangent bundle.