Modern Geometry I: Problem Set 2

Due Monday, October 18

Problem 1: Show that the Lie bracket of vector fields satisfies the Jacobi identity, i.e. for three vector fields X_1, X_2, X_3

$$[[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] = 0$$

Problem 2:

Let $O(n) = \{A \in M_n(\mathbf{R}) | AA^t = I\}$ be the space of orthogonal n by $n$matrices. Define a function

$$f : M_n(\mathbf{R}) \to M_n(\mathbf{R})$$

by

$$f(A) = AA^t$$

so that $O(n) = f^{-1}(I)$.
a) show that f is a smooth function and that its derivative is given by

$$f_*(A)V = VA^t + AV^t$$

b) use this to show that $O(n)$ is a smooth submanifold of $\mathbf{R}^{n^2}$.

Problem 3:

Given a smooth map $f : M \to N$ between smooth manifolds M and N, show that the pull-back map $f^* : \Omega^*(N) \to \Omega^*(M)$ satisfies

a) $f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$ for $\omega_1, \omega_2 \in \Omega^*(N)$

b) $f^*d = df^*$

Problem 4: Show that, for M a smooth manifold, $X_1, \ldots, X_{k+1}$ smooth vector fields on M, and $\omega \in \Omega^k(M)$

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{k+1} X_i(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \end{aligned}$$

Problem 5: Prove the two Cartan formulas relating the Lie derivative L_X, the interior product i_X and the exterior derivative d

a) $L_X = di_X + i_X d$

b) $L_X i_Y - i_Y L_X = i_{[X,Y]}$

Problem 6:

Using the definition of the Poisson bracket of two functions in terms of a symplectic 2-form ω:

$$\{f, g\} = -\omega(X_f, X_g)$$

(where X_f is determined by $i_{X_f}\omega = -df$)
show that it satisfies the Jacobi identity, i.e. for three functions f_1, f_2, f_3 one has

$$\{\{f_1, f_2\}, f_3\} + \{\{f_3, f_1\}, f_2\} + \{\{f_2, f_3\}, f_1\}$$