Groups and Representations II: Final Exam (REVISED) **Due Monday, May 12**

Problem 1: Show that if λ is a dominant weight (i.e. in the closure of the fundamental Weyl chamber), and ω is a fundamental weight, then the dimension of the irreducible representation $V_{\lambda+\omega}$ is greater than the dimension of V_λ. Use this to show that the non-trivial representations of smallest dimension must be among the fundamental representations.

Problem 2: For a compact Lie group G, use the Killing form to define the Laplacian Δ on G. This is the left and right-invariant second order differential operator on G corresponding to the quadratic Casimir operator defined in the first problem set. Let W_λ be the linear span of the space of matrix elements of an irreducible representation with highest weight λ. Show that W_λ is an eigenspace of Δ with eigenvalue given by

$$||\lambda+\delta||^2 - ||\delta||^2$$

where δ is half the sum of the positive roots.

Problem 3: Consider the group $G = SO(4)$ and its double cover $Spin(4)$. For both of these groups:

1. Identify the maximal torus, the roots, the Weyl group and a choice of a set of positive roots.
2. Classify the irreducible representation of $SO(4)$ and of $Spin(4)$. What are the weights in each of these representations?

Problem 4: For any irreducible representation representation V_λ of a compact Lie group G (of highest weight λ), define an irreducible representation on its dual space V_λ^*. How are the weights of V_λ^* related to those of V_λ? For the case $G = SU(3)$, representations are characterized by the integer coefficients (m_1, m_2) of their highest weight, expressed in terms of the fundamental weights. What are the coefficients for V_λ^* if they are (m_1, m_2) for V_λ. For the representation labelled $(2, 0)$, what are the weights that occur in this representation? What about in the dual representation?

Problem 5: Prove the Kostant multiplicity formula. This says that the multiplicity of the weight μ in the representation of highest weight λ is given by

$$\sum_{w\in W} sgn(w)P(w(\lambda+\delta)-(\mu+\delta))$$

where W is the Weyl group, δ is half the sum of the positive roots, and $P(\lambda)$ is the number of ways to write λ as a sum of positive roots with non-negative coefficients.

Problem 6: Classify the irreducible representations of S_4, and use the Frobenius character formula to compute the table of values of the character for these representations.

Problem 7: Using the Jacobi identity, derive the "cocycle" condition that an antisymmetric two-form ω on $\mathfrak{g}$ must satisfy in order to be used to define a central extension of the $\mathfrak{g}$ by $\mathbf{C}$. Show that for $\mathfrak{g} = Vect_{\mathbf{C}}(S^1)$

$$\omega(e^{in\theta}\frac{d}{d\theta}, e^{im\theta}\frac{d}{d\theta}) = \delta_{n,-m}\frac{m(m^2-1)}{12}$$

satisfies the cocycle condition.