BRST and Dirac Cohomology
Preliminary Draft Version

Peter Woit
Department of Mathematics, Columbia University
woit@math.columbia.edu

July 11, 2009

1 Introduction

Given a quantum mechanical system with symmetry group a Lie group G, the Hilbert space $\mathcal{H}$ of the theory will be a unitary representation of G and of its Lie algebra $\mathfrak{g}$. For the case of G a gauge symmetry, the space of physical states is supposed to be the G-invariant subspace

$$\mathcal{H}^G \subset \mathcal{H}$$

The quantum BRST method constructs this subspace using what mathematicians refer to as Lie algebra cohomology, in which

$$\mathcal{H}^G = H^0(\mathfrak{g}, \mathcal{H})$$

an identity which follows from the abstract definition of such a cohomology theory as the derived functor of the invariants functor. Lie algebra cohomology can be explicitly defined in terms of a specific complex with differential, the Chevalley-Eilenberg complex

$$C^*(\mathfrak{g}, \mathcal{H}) = \mathcal{H} \otimes \Lambda^*(\mathfrak{g}^*)$$

In physicist's language, one adjoins to the original theory anti-commuting variables transforming as $\mathfrak{g}$, known as "ghosts".

For a more useful characterization of such a quantum system, it should be given as a "quantization" of a classical mechanical system described by an algebra of functions $C^\infty(P)$ on a Poisson manifold P, with a Poisson bracket $\{\cdot, \cdot\}$. Quantization takes such functions to operators on $\mathcal{H}$, and (at least for generators of $C^\infty(P)$), Poisson brackets to commutators. When G is a symmetry group of P, one defines a "reduced" Poisson manifold $P//G$, and quantization of this space is supposed to give operators on the quantum reduction $\mathcal{H}^G$. The operations of reduction and quantization should commute.

The Hamiltonian BRST method provides a homological "classical BRST" construction of functions on $P//G$ in terms of a complex with differential given by taking the super-Poisson bracket with a specific element Ω of the complex. Quantization of Ω then provides an explicit differential, the "BRST operator" Q. The physical Hilbert space $\mathcal{H}^G$ should be the degree zero cohomology of Q acting on states, and physical operators should be given by degree zero cohomology classes of Q acting on operators (as $\mathcal{O} \to [Q, \mathcal{O}]$).

Attempts to implement this method run into various technical problems, due, for example, to the lack of a quantization map with the properties one needs, to non-zero higher cohomology classes, or to the necessity of introducing an indefinite inner product on states. To better understand the mathematical issues involved, one can study a special case of the problem, that of $P = \mathfrak{g}^*$ (this is a Poisson manifold with G symmetry). A simple case of the BRST formalism would be to choose as gauge symmetry group some $H \subset G$. This will be reviewed in the first section of the paper.

It turns out that if one adopts a somewhat different point of view on the topic of quantum gauge symmetries, then a very interesting mathematical formalism somewhat analogous to the BRST formalism exists, one that has been recently intensively studied by representation theorists under the name "Dirac cohomology". Here, given a symmetry group G, instead of choosing a subgroup H to gauge, one chooses a subgroup R with Lie algebra $\mathfrak{r}$ that will remain ungauged. An algebraic Dirac operator $\not{\partial}_{\mathfrak{g}/\mathfrak{r}}$ plays the role of the BRST operator, although its square is no longer zero, but a central element. As a result it acts as a differential on operators, with cohomology that turns to be the algebra $Z(\mathfrak{r})$, the center of the enveloping algebra $U(\mathfrak{r})$.

On states the kernel of $\not{\partial}_{\mathfrak{g}/\mathfrak{r}}$ gives an analog of BRST-invariant states, while allowing a positive-definite inner product. When the quotient $\mathfrak{g}/\mathfrak{r}$ can be decomposed as $\overline{\mathfrak{u}} \oplus \mathfrak{u}$ for some complex Lie algebra $\mathfrak{u}$, these states are essentially harmonic representatives of the Lie algebra cohomology $H^*(\mathfrak{u}, \mathcal{H})$. In the standard BRST method, this is what would appear when gauging the complex Lie group with Lie algebra $\mathfrak{u}$. Note that this is a sort of Gupta-Bleuler construction, gauging $\mathfrak{u}$, but not all of $\mathfrak{g}/\mathfrak{r} = \overline{\mathfrak{u}} \oplus \mathfrak{u}$. The Dirac cohomology construction can be applied in cases where the the BRST method cannot: there is no need for $\mathfrak{g}/\mathfrak{r}$ to either be a Lie algebra, or to decompose as a sum of a Lie algebra and its conjugate.

Dirac cohomology gives a version of the standard highest-weight theory for representations of semi-simple Lie algebras, as well as other applications in representation theory. For an exposition of some of these applications, see the recent book by Huang and Pandzic[12]. After reviewing the specifics of the Dirac cohomology construction and examining some examples, it will be applied to handle the gauge symmetry in the toy model of 0+1 dimensional gauge theory.

In 1+1 dimensions, there may be applications to geometric Langlands theory, and in higher dimensions the use of Dirac cohomology instead of BRST may conceivably lead to new insights into both gauge theory and the physical Dirac operator.

2 Conventional Hamiltonian BRST

In the conventional constrained Hamiltonian formalism, one begins with a classical mechanical system characterized by a Poisson algebra $\mathcal{P}$, with Poisson bracket $\{\cdot,\cdot\}$. $\mathcal{P}$ is a Lie algebra with the Poisson bracket a Lie bracket.

First-class constraints give an ideal $I \subset \mathcal{P}$, closed under the Poisson bracket. $\phi \in I$ acts on the quotient A/I by

$$\phi \cdot [f] = [\{\phi, f\}]$$

and one can define the reduced algebra by taking I invariants

$$\mathcal{P}_{red} = (A/I)^I$$

This is again a Poisson algebra, the algebra of physical observables.

Quantization is supposed to give a map $q : \mathcal{P} \to \mathcal{A}$ taking Poisson algebras to associative algebras. Such a map should be a Lie algebra homomorphism modulo higher order terms in algebra generators (the Lie bracket for associative algebras is the commutator). For a Poisson algebra $\mathcal{P}$, quantization gives an algebra of operators $q(\mathcal{P})$, represented as linear operators on a Hilbert space $\mathcal{H}$.

In the presence of constraints, physical states are supposed to be those in the subspace $\mathcal{H}_{red} \subset \mathcal{H}$ annihilated by operators in $q(I)$. Physical observables correspond to operators in the algebra $q(\mathcal{P}_{red})$, which is supposed to be an algebra of operators on $\mathcal{H}_{red}$.

The basic idea of the BRST formalism (a good reference is [9]) is to first use techniques from homological algebra to enlarge $\mathcal{P}$ to a graded super-Poisson algebra $\mathcal{P}^*_{BRST}$ with super-Poisson bracket $\{\cdot,\cdot\}_\pm$ and differential

$$D(\cdot) = \{\Omega, \cdot\}_\pm$$

for some $\Omega \in \mathcal{P}^*_{BRST}$. D should satisfy $D^2 = 0$ and the cohomology of the complex in degree zero is arranged to be

$$\mathcal{P}_{red} = H^0(\mathcal{P}^*_{BRST})$$

This construction is sometimes referred to as "classical BRST".

One then generalizes the quantization map q to take super-Poisson algebras to associative super-algebras and thereby get a quantum version of BRST. This involves a graded algebra of operators $q(\mathcal{P}^*_{BRST})$ acting on a graded Hilbert space $\mathcal{H}^*_{BRST}$, with a distinguished degree one operator $Q = q(\Omega)$ (the "BRST operator") satisfying $Q^2 = 0$. The physical Hilbert space of states is given by the degree zero cohomology

$$\mathcal{H}_{red} = H^0_Q(\mathcal{H}^*_{BRST})$$

of the complex $\mathcal{H}^*_{BRST}$ with differential Q. The algebra of physical operators acting on these states is given by

$$q(\mathcal{P}_{red}) = H^0_d(q(\mathcal{P}^*_{BRST}))$$

the degree zero cohomology of the complex $q(\mathcal{P}^*_{BRST})$, with differential given by $d(\cdot) = [Q, \cdot]_\pm$.

For any given choice of $\mathcal{P}$ and I, one runs into various problems with implementing this philosophy, problems which one may or may not be able to successfully resolve.

More concretely, the Poisson algebra of interest will typically be an algebra $\mathcal{P} = C^\infty(P)$, functions on a Poisson manifold P, for instance $\mathcal{P} = C^\infty(P)$ or $\mathcal{P} = Pol(P)$ (polynomials on P). Such a manifold P may be a symplectic manifold, but more generally it will be a manifold foliated by symplectic leaves. An important example will be $P = \mathfrak{g}^*$, the dual space to the Lie algebra $\mathfrak{g}$, with $\mathcal{P} = Pol(\mathfrak{g}^*) = S^*(\mathfrak{g})$ ($S^*(\mathfrak{g})$ is the symmetric algebra on $\mathfrak{g}$). In that case, the Lie bracket directly gives a Poisson bracket, since linear functions on $\mathfrak{g}^*$ are elements of $\mathfrak{g}$. So, one can define the Poisson bracket of $f_1, f_2 \in Pol(\mathfrak{g}^*)$ at $\xi \in \mathfrak{g}^*$ by

$$\{f_1, f_2\}(\xi) = \xi([df_1, df_2])$$

If P is a Poisson manifold with symmetry group G and $\mathfrak{g} = Lie(G)$, G acts on P preserving the Poisson bracket. This situation that can alternately be characterized by the existence of a Poisson map between Poisson manifolds, the moment map

$$\mu : P \to \mathfrak{g}^*$$

In this case, the ideal of first-class constraints is the ideal of functions $f \in C^\infty(P)$ that vanish on $\mu^{-1}(0)$. The reduced space is $P_{red} = \mu^{-1}(0)/G$, sometimes written $P//G$. One would like to define the reduced Poisson algebra as $C^\infty(P//G)$, but one problem with this is that $P//G$ is often a singular space. If 0 is a regular value for μ, the constraints are said to be irreducible.

In the irreducible case, the classical BRST construction proceeds by first using the moment map to construct a complex $C^\infty(P) \otimes \Lambda^*(\mathfrak{g})$ with differential

$$\delta : C^\infty(P) \otimes \Lambda^*(\mathfrak{g}) \to C^\infty(P) \otimes \Lambda^{*-1}(\mathfrak{g})$$

The dual of the moment map takes $\mathfrak{g}$ to $C^\infty(P)$, and this can be extended to a homomorphism of the symmetric algebra $S^*(\mathfrak{g})$ to $C^\infty(P)$. This makes $C^\infty(P)$ a $S^*(\mathfrak{g})$-module, and our complex is the Koszul resolution of this module. Its homology is just the functions on $\mu^{-1}(0)$ in degree zero:

$$H^0_\delta(C^\infty(P) \otimes \Lambda^*(\mathfrak{g})) = C^\infty(\mu^{-1}(0))$$

One then takes G invariants by taking the Lie algebra cohomology of this in degree zero, which requires tensoring the complex by $\Lambda^*(\mathfrak{g}^*)$ and defining the Chevalley-Eilenberg differential d. One ends up with a double complex

$$C^\infty(P) \otimes \Lambda^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g}^*)$$

with a total differential D. Kostant and Sternberg[19] show that under certain conditions which are not necessarily satisfied (vanishing of the higher $Tor_*^{S^*(\mathfrak{g})}(\mathbf{R}, C^\infty(P))$

$$H^0_D(C^\infty(P) \otimes \Lambda^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g}^*))$$

is the reduced Poisson algebra $C^\infty(P//G)$. So, in this case D is the classical BRST operator and

$$\mathcal{P}^*_{BRST} = C^\infty(P) \otimes \Lambda^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g}^*)$$

Kostant and Sternberg go on to use the fact that the super-algebras $\Lambda^*(\mathfrak{g}) \otimes \Lambda^*(\mathfrak{g}^*)$ and $\Lambda^*(\mathfrak{g}+\mathfrak{g}^*)$ are isomorphic, and they identify an element $\Omega \in C^\infty(P) \otimes \Lambda^*(\mathfrak{g}+\mathfrak{g}^*)$ such that

$$D(\cdot) = \{\Omega, \cdot\}_\pm$$

(this uses a super-Poisson bracket $\{\cdot,\cdot\}_\pm$ on $C^\infty(P) \otimes \Lambda^*(\mathfrak{g}+\mathfrak{g}^*)$), completing the classical BRST construction.

To get a quantum version of this, Kostant and Sternberg first assume as given a quantization $q(C^\infty(P))$ as operators on a Hilbert space $\mathcal{H}$, and quantize $\Lambda^*(\mathfrak{g}+\mathfrak{g}^*)$ as the Clifford algebra $Cliff(\mathfrak{g}+\mathfrak{g}^*)$. This Clifford algebra is defined using the inner product coming from evaluating element of $\mathfrak{g}^*$ on elements of $\mathfrak{g}$. It has a unique irreducible module, the spin module S, which for a unimodular group G is isomorphic to $\Lambda^*(\mathfrak{g})$.

In summary, the Kostant-Sternberg version of classical BRST is to take

$$\mathcal{P}^*_{BRST} = \mathcal{P} \otimes \Lambda^*(\mathfrak{g}+\mathfrak{g}^*)$$

with differential $d(\cdot) = \{\Omega, \cdot\}_\pm$. Quantum BRST for operators uses the complex

$$q(\mathcal{P}) \otimes Cliff(\mathfrak{g}+\mathfrak{g}^*)$$

with a distinguished operator $Q = q(\Omega)$ such that the differential is $d(\cdot) = [Q, \cdot]$. Q satisfies $Q^2 = 0$ and also acts on states in $\mathcal{H}_{BRST} = \mathcal{H} \otimes S$, with cohomology classes the physical states, acted on by cohomology classes of operators.

For P a point, so $\mathcal{P} = \mathbf{C}$, and $\mathfrak{g}$ semi-simple, $\Omega \in \Lambda^3(\mathfrak{g})$ is defined using the Killing form. The quantum BRST operator cohomology is trivial, the complex Q acting on $\mathcal{H}_{BRST}$ is the Chevalley-Eilenberg complex for $H^*(\mathfrak{g}, \mathbf{C})$, so the quantum BRST state cohomology is the cohomology of the Lie algebra.

Sevostyanov[23] gives an interpretation of the Kostant-Sternberg BRST operator cohomology as a homological version

$$Hecke^*(A, U(\mathfrak{g}))$$

of the Hecke algebra associated to a pair of algebras $U(\mathfrak{g}) \subset A$, for the case $A = q(\mathcal{P})$. Here

$$Hecke^0(A, U(\mathfrak{g})) = End_A(A \otimes_{U(\mathfrak{g})} \mathbf{C}, A \otimes_{U(\mathfrak{g})} \mathbf{C})$$

which acts on the Lie algebra cohomology $H^*(\mathfrak{g}, \mathcal{H})$.

2.1 An Example

Feigin and Frenkel[4] give an interesting example of the above construction. Take as Poisson manifold $P = \mathfrak{g}^*$ for $\mathfrak{g}$ a complex semi-simple Lie algebra, so the

Poisson algebra is $\mathcal{P} = S^*(\mathfrak{g})$. This can be quantized with the symmetrization map

$$q : S^*(\mathfrak{g}) \to U(\mathfrak{g})$$

and one can take as Hilbert space $\mathcal{H}$ any representation of $\mathfrak{g}$.

Note that $U(\mathfrak{g})$ is a filtered algebra, with $S^*(\mathfrak{g})$ its associated graded algebra. The Poisson bracket on $S^*(\mathfrak{g})$ is induced from the commutator on $U(\mathfrak{g})$, and the quantization map q takes the Poisson bracket not to the commutator, but to the commutator plus lower order terms.

Such a Lie algebra has a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$$

where $\mathfrak{h}$ is the Cartan subalgebra, $\mathfrak{n}$ the positive root spaces and $\overline{\mathfrak{n}}$ the negative root spaces. One can choose to treat $\mathfrak{n}$ as a gauge symmetry, with gauge group N such that $Lie(N) = \mathfrak{n}$. In this case one must reduce not at the trivial representation of N, but at the principal character $\chi \in \mathfrak{n}^*$. The moment map

$$\mu : \mathfrak{g}^* \to \mathfrak{n}^*$$

is just the projection map, and the reduced Poisson manifold is

$$\mathfrak{g}^*//N = \mu^{-1}(\chi)/N$$

The classical BRST construction gives the Poisson algebra on this space as the cohomology of a complex

$$S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{n} \oplus \overline{\mathfrak{n}})$$

Quantum BRST for operators uses a complex

$$U(\mathfrak{g}) \otimes Cliff(\mathfrak{n} \oplus \overline{\mathfrak{n}})$$

which, by a theorem of Kostant[16] has cohomology purely in degree zero, isomorphic to the center $Z(\mathfrak{g})$. This acts on the BRST state cohomology, which consists of so-called Whittaker vectors $v \in \mathcal{H}$ satisfying $Xv = \chi(X)v$ for $X \in \mathfrak{n}$.

Feigin and Frenkel go on to apply this in the infinite-dimensional case of $\mathfrak{g}$ an affine Kac-Moody algebra, where gauging a nilpotent sub-algebra is Drinfeld-Sokolov reduction. Here the BRST method requires the use of the semi-infinite variant of Lie algebra cohomology, and the reduced algebra is known as a W-algebra. This sort of construction of the W-algebra gets used in one approach to the geometric Langlands program.

3 The Quantum Weil Algebra

A semi-simple Lie algebra $\mathfrak{g}$ comes with an invariant bilinear form $B(\cdot,\cdot)$, the Killing form. This can be used to identify $\mathfrak{g} = \mathfrak{g}^*$, and thus the symmetric algebras $S^*(\mathfrak{g})$ and $S^*(\mathfrak{g}^*)$. We have seen that the quantization $q(S^*(\mathfrak{g}))$ is the universal enveloping algebra $U(\mathfrak{g})$, and B allows the same definition to be used to quantize $S^*(\mathfrak{g}^*)$, the polynomials on $\mathfrak{g}$. The invariant bilinear form can also be quantize the anti-commuting analog of $S^*(\mathfrak{g}^*)$, the exterior algebra $\Lambda^*(\mathfrak{g}^*)$, which can be thought of as functions on a space $\mathfrak{g}$ of anti-commuting variables. This quantization gives the Clifford algebra $Cliff(\mathfrak{g})$.

The Weil algebra $W^*(\mathfrak{g})$ is defined to be the tensor product

$$W^*(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*)$$

It is a graded algebra with generators of $S^*(\mathfrak{g}^*)$ carrying degree two, generators of $\Lambda^*(\mathfrak{g}^*)$ degree one. It is a differential graded algebra, with a differential d that makes it cohomologically trivial, i.e. $H^0(W^*(\mathfrak{g})) = \mathbf{C}$ (for more details on the Weil algebra, a good reference is the book of Guillemin and Sternberg[8].

If $\mathfrak{g}$ is the Lie algebra of a Lie group G, and P is a principal G-bundle over a manifold M, then a connection on P corresponds to an equivariant map

$$\theta : \mathfrak{g}^* \to \Omega^*(P)$$

($\Omega^*(P)$ is the de Rham complex of differential forms on P). This can be extended to a homomorphism of differential algebras

$$\theta : W^*(\mathfrak{g}) \to \Omega^*(P)$$

which takes the generators of $\Lambda^*(\mathfrak{g}^*)$ to connection one-forms on P, the generators of $S^*(\mathfrak{g}^*)$ to curvature two-forms.

$W(\mathfrak{g}^*)$ and $\Omega^*(P)$ are not just differential graded algebras, but are $\mathfrak{g}$-differential graded algebras. This means their is an action on them of, for each $X \in \mathfrak{g}$, Lie derivative operators $\mathcal{L}_X$ of degree zero, and contraction operators of degree -1, satisfying certain compatibility conditions, including the Cartan relation

$$di_X + i_X d = \mathcal{L}_X$$

In each case one can define a basic sub-complex of the algebra, the sub-complex annihilated by these operators. One has $(\Omega^*(P))^{basic} = \Omega^*(M)$, and $(W^*(\mathfrak{g}))^{basic} = S^*(\mathfrak{g}^*)^{\mathfrak{g}}$, the invariant polynomials on $\mathfrak{g}$.

Restricting to basic sub-complexes, a connection gives a homomorphism

$$\theta : S^*(\mathfrak{g}^*)^{\mathfrak{g}} \to \Omega^*(M)$$

This is the Chern-Weil homomorphism which takes invariant polynomials on $\mathfrak{g}$ to differential forms on M constructed out of curvature two-forms. Taking cohomology, for compact G this homomorphism is independent of the connection and gives invariants of the bundle in $H^*(M)$. In this case one can think of the complex $W^*(\mathfrak{g})$ as a subcomplex of the de Rham complex $\Omega^*(EG)$ of a homotopically trivial space EG with free G-action, one that carries its cohomology, just as the Chevalley-Eilenberg complex is a subcomplex of $\Omega^*(G)$ carrying the cohomology of the manifold G. For the classifying space $BG = EG/G$, its cohomology ring can be computed as the cohomology of $W^*(\mathfrak{g})^{basic}$, which, since the differential is trivial, is just the ring of invariant polynomials $S^*(\mathfrak{g}^*)^{\mathfrak{g}}$.

Alekseev and Meinrenken[1] (for some related expository material, see [21]) define a quantization map

$$q : W^*(\mathfrak{g}) = S^*(\mathfrak{g}^*) \otimes \Lambda^*(\mathfrak{g}^*) \rightarrow \mathcal{W}(\mathfrak{g}) = U(\mathfrak{g}) \otimes Cliff(\mathfrak{g})$$

On the two factors this restricts to the separate quantization maps discussed earlier, but the full map q is not just the tensor product map. Here $\mathcal{W}(\mathfrak{g})$ is called the quantum (or non-commutative) Weil algebra. It carries a $\mathbf{Z}$-filtration, but is just $\mathbf{Z}_2$-graded, not $\mathbf{Z}$-graded. q is a homomorphism of $\mathfrak{g}$-differential graded algebras, intertwining the actions of the operators $d, \mathcal{L}_X, i_X$ on both algebras. Restricted to the basic subalgebras it becomes the Duflo map

$$q : S^*(\mathfrak{g})^{\mathfrak{g}} \rightarrow U(\mathfrak{g})^{\mathfrak{g}} = Z(\mathfrak{g})$$

which is an ismorphism.

On $\mathcal{W}(\mathfrak{g})$, $d, \mathcal{L}_X, i_X$ act by inner automorphisms, i.e. (super)-commutation with specific elements of $\mathcal{W}(\mathfrak{g})$. In particular, d is given by

$$d(\cdot) = [\not{D}_{\mathfrak{g}}, \cdot]_{\pm}$$

for a special element $\not{D}_{\mathfrak{g}}$ which can be thought of as a sort of algebraic Dirac operator, with a cubic term. It has some remarkable properties, one of which is that it can be defined by

$$\not{D}_{\mathfrak{g}} = q(CS)$$

i.e as the quantization of the Chern-Simons element $CS \in W(g)$. Given a connection $\theta : W(g) \rightarrow \Omega^*(P)$, CS is the element that maps to the Chern-Simons form of the connection. It satisfies $d(CS) = C_2$, where C_2 is the quadratic element of $S^*(\mathfrak{g}^*)$ constructed from the Killing form.

This way of defining $\not{D}_{\mathfrak{g}}$ appears in [2], for a more explicit definition, see [1]. The same operator appears earlier in the physics literature as the supersymmetry generator for the superparticle on the group G (see, e.g. [6]).

Another remarkable property of $\not{D}_{\mathfrak{g}}$ is the formula for its square. Using

$$\not{D}_{\mathfrak{g}}^2 = \frac{1}{2}[\not{D}_{\mathfrak{g}}, \not{D}_{\mathfrak{g}}]_{\pm} = \frac{1}{2}d(\not{D}_{\mathfrak{g}}) = \frac{1}{2}dq(CS) = \frac{1}{2}q(dCS) = \frac{1}{2}q(C_2)$$

and the formula for the Duflo map, one finds that

$$\not{D}_{\mathfrak{g}}^2 = -\Omega_{\mathfrak{g}} - \frac{1}{24}tr_{\mathfrak{g}}(\Omega_{\mathfrak{g}})$$

where $\Omega_{\mathfrak{g}}$ is the quadratic Casimir element in $U(\mathfrak{g})$. Thus $\not{D}_{\mathfrak{g}}^2 \in Z(\mathfrak{g})$, so it commutes with everything and $d^2 = 0$ on $\mathcal{W}(\mathfrak{g})$.

The cohomology of $\mathcal{W}(\mathfrak{g})$ with respect to this differential d is just the constants. On an irreducible representation of highest weight λ,

$$\not{D}_{\mathfrak{g}}^2 = -|\lambda + \rho_{\mathfrak{g}}|^2 Id$$

where $\rho_{\mathfrak{g}}$ is half the sum of the positive roots. This is negative-definite, so $ker\, \not{D}_{\mathfrak{g}} = \emptyset$ for the action of $D_{\mathfrak{g}}$ on $V \otimes S$, for any representation V.

4 Dirac Cohomology for Quadratic Subalgebras

The quantum Weil algebra of the previous section is a special case of a more general construction given by Kostant[18] in 2002. The more general context is

- $\mathfrak{g}$ a complex semi-simple Lie algebra.
- $B(\cdot,\cdot)$ a non-degenerate, invariant symmetric bilinear form on $\mathfrak{g}$ (e.g. the Killing form).
- $\mathfrak{r}$ a reductive subalgebra of $\mathfrak{g}$ on which $B(\cdot,\cdot)$ is non-degenerate.
- V a representation of $\mathfrak{g}$, or equivalently, a module for the universal enveloping algebra $U(\mathfrak{g})$.

In such a situation one can define

- $\mathfrak{s}$ is the orthocomplement to $\mathfrak{r}$ with respect to B. One has $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ as vector spaces and $[\mathfrak{r}, \mathfrak{s}] \subset \mathfrak{s}$. In general $\mathfrak{s}$ is just a vector subspace, it is not a Lie sub-algebra of $\mathfrak{g}$.
- $Cliff(\mathfrak{s})$ is the Clifford algebra associated to $\mathfrak{s}$ and the inner product $B(\cdot,\cdot)|_{\mathfrak{s}}$.
- S is a spinor module for $Cliff(\mathfrak{s})$. For $\mathfrak{s}$ even (complex) dimensional, this is unique up to isomorphism.

4.1 The Vogan Conjecture and the Kostant Dirac Operator

Given a subgroup $R \subset G$ with Lie algebra $\mathfrak{r}$, the space EG has a free action of R, and $EG/R = BR$. The Weil algebra $W(\mathfrak{g})$ provides a tractable algebraic model for the de Rham cohomology of EG, and the subalgebra $W(\mathfrak{g})^{\mathfrak{r}-basic}$ of basic elements for the $\mathfrak{r}$ action (annihilated by $\mathcal{L}_X,\ i_X$, for $X \in \mathfrak{r}$) provides a model for the deRham cohomology of BR. One has

$$H^*(W(\mathfrak{g})^{\mathfrak{r}-basic}) = H^*(BR) = S^*(\mathfrak{r}^*)$$

In the quantum Weil algebra case, one has

$$H^*(\mathcal{W}(\mathfrak{g})^{\mathfrak{g}-basic}) = Z(\mathfrak{g})$$

and expects that

$$H^*(\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}) = Z(\mathfrak{r})$$

In a more explicit form, this sort of statement was first conjectured by Vogan[24], then proved by Huang-Pandzic[10], and more generally, by Kostant[18]. For a more abstract proof, see Alekseev-Meinrenken[2] and [20].

To make this isomorphism more explicit, note first that

$$\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic} = (U(\mathfrak{g}) \otimes Cliff(s))^{\mathfrak{r}}$$

the $\mathfrak{r}$-invariant sub-complex of $U(\mathfrak{g}) \otimes Cliff(s)$. $\mathfrak{r}$ acts "diagonally" here, not just on the $U(\mathfrak{g})$ factor. In other words, we are using the homomorphism

$$\zeta : U(\mathfrak{r}) \to U(\mathfrak{g}) \otimes Cliff(\mathfrak{s})$$

defined by

$$\zeta(X) = X \otimes 1 + 1 \otimes \nu(X)$$

where

$$\nu : \mathfrak{r} \to Lie(SO(\mathfrak{s})) \subset Cliff(\mathfrak{s})$$

is the representation of $\mathfrak{r}$ on the spinor module S as quadratic elements in $Cliff(\mathfrak{s})$ coming from the fact that the adjoint action of $\mathfrak{r}$ on $\mathfrak{s}$ is an orthogonal action.

The Kostant Dirac operator is defined as the difference of the algebraic Dirac operators for $\mathfrak{g}$ and $\mathfrak{r}$

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = \not{D}_{\mathfrak{g}} - \not{D}_{\mathfrak{r}}$$

It is an element of $\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}$ and the restriction of the differential d on $\mathcal{W}(\mathfrak{g})$ to $\mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic}$ is given by

$$d(\cdot) = [\not{D}_{\mathfrak{g},\mathfrak{r}}, \cdot]$$

The map

$$Z(\mathfrak{r}) = U(\mathfrak{r})^{\mathfrak{r}} \to \mathcal{W}(\mathfrak{g})^{\mathfrak{r}-basic} = (U(\mathfrak{g}) \otimes Cliff(s))^{\mathfrak{r}}$$

takes values on cocycles for d and is an isomorphism on cohomology.

Even more explicitly, one can write

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = \sum_{i=1}^{n} Z_i \otimes Z_i + 1 \otimes v$$

where

$$v = \frac{1}{2} \sum_{1 \leq i,j,k \leq n} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k$$

and the Z_i are an orthonormal basis of $\mathfrak{s}$.

The square of the Kostant Dirac operator is given by

$$\not{D}^2_{\mathfrak{g},\mathfrak{r}} = -\Omega_{\mathfrak{g}} \otimes 1 + \zeta(\Omega_{\mathfrak{r}}) + (|\rho_{\mathfrak{r}}|^2 - |\rho_{\mathfrak{g}}|^2) 1 \otimes 1$$

where $\rho_{\mathfrak{g}}$ is half the sum of the positive roots for $\mathfrak{g}$, $\rho_{\mathfrak{r}}$ the same for $\mathfrak{r}$.

4.2 Dirac Cohomology: Operators

$\not{D}^2_{\mathfrak{g},\mathfrak{r}}$ commutes with all elements of $U(\mathfrak{g}) \otimes Cliff(\mathfrak{s})$ so the differential d satisfies $d^2 = 0$. The differential d is an equivariant map for the $U(\mathfrak{r})$ action given by ζ, so it is also a differential on $(U(\mathfrak{g}) \otimes Cliff(\mathfrak{s}))^{\mathfrak{r}}$. The operator Dirac cohomology is defined as

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}) = Ker\ d / Im\ d$$

on $(U(\mathfrak{g}) \otimes Cliff(\mathfrak{s}))^{\mathfrak{r}}$. The Vogan conjecture says that it is isomorphic to $Z(\zeta(\mathfrak{r}))$. Note that in general this will be not $\mathbf{Z}$-graded, but just $\mathbf{Z}_2$-graded, using the $\mathbf{Z}_2$ grading of $Cliff(\mathfrak{s})$).

So, for any representation V of $\mathfrak{g}$, we have an algebra of operators $(U(\mathfrak{g}) \otimes Cliff(\mathfrak{s}))^{\mathfrak{r}}$ acting on $V \otimes S$, with differential $d = ad(\not{D}_{\mathfrak{g},\mathfrak{r}})$ and cohomology isomorphic to the center $Z(\mathfrak{r})$. It is this algebra that will play the role of the algebra of BRST-invariant operators in this analog of BRST.

4.3 Dirac Cohomology: States

$\not{D}_{\mathfrak{g},\mathfrak{r}}$ acts on $V \otimes S$, and one can define the state Dirac cohomology as

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\, \not{D}_{\mathfrak{g},\mathfrak{r}}/(Im\, \not{D}_{\mathfrak{g},\mathfrak{r}} \cap Ker\, \not{D}_{\mathfrak{g},\mathfrak{r}})$$

Since $\not{D}^2_{\mathfrak{g},\mathfrak{r}} \neq 0$, this is not a standard sort of homological differential. In particular, one has no assurance in general that

$$Im\, \not{D}_{\mathfrak{g},\mathfrak{r}} \subset Ker\, \not{D}_{\mathfrak{g},\mathfrak{r}}$$

However, if V is either finite-dimensional or a unitary representation, then an inner product on $V \otimes S$ can be chosen so that $\not{D}_{\mathfrak{g},\mathfrak{r}}$ will be skew self-adjoint. In that case $Ker(\not{D}_{\mathfrak{g},\mathfrak{r}}) = Ker(\not{D}^2_{\mathfrak{g},\mathfrak{r}})$ and one has

$$H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\, \not{D}_{\mathfrak{g},\mathfrak{r}}$$

For an irreducible representation V of $\mathfrak{g}$, a well-known invariant is the infinitesimal character $\chi(V)$. Such infinitesimal characters can be identified with orbits in $\mathfrak{h}^*$ ($\mathfrak{h}$ is a Cartan sub-algebra) under the Weyl group $W_{\mathfrak{g}}$, with a representation of highest weight $\lambda \in \mathfrak{h}^*$ corresponding to the orbit of $\lambda + \delta_{\mathfrak{g}}$. The Dirac cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V)$ of a representation V also provides an invariant of the representation V. For a finite-dimensional V, it will consist of a collection of $|W_{\mathfrak{g}}|/|W_{\mathfrak{r}}|$ $\mathfrak{r}$ irreducibles. These will all have the same infinitesimal character as V, when one includes $\mathfrak{t}^* \subset \mathfrak{h}^*$ ($\mathfrak{t}$ is the Cartan sub-algebra of $\mathfrak{r}$) by extending element of $\mathfrak{t}^*$ as zero on $\mathfrak{h}/\mathfrak{t}$. This phenomenon was first noticed by Pengpan and Ramond for the case $G = F_4$ and $R = Spin(9)$ and was explained in general in [7], which led to Kostant's discovery[17] of his version of the Dirac operator.

5 Dirac Cohomology and Lie algebra Cohomology

The BRST method uses an operator Q satisfying Q^2, and describes states in terms of Lie algebra cohomology of the gauged subgroup, whereas the Dirac cohomology construction described above appears to be rather different. It depends on a choice of un-gauged subgroup R and defines states as the kernel of an operator that does not square to zero. It turns out though that these two methods give essentially the same thing in the case that

$$\mathfrak{g}/\mathfrak{r} = \mathfrak{u} \oplus \overline{\mathfrak{u}}$$

In this case, $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are isotropic subspaces with respect to the symmetric bilinear form B, and one can identify $\mathfrak{u}^* = \overline{\mathfrak{u}}$. The spinor module S for $Cliff(\mathfrak{s})$ can be realized explicitly on either $\Lambda^*(\mathfrak{u})$ or on $\Lambda^*(\overline{\mathfrak{u}})$. However, when one does this, the adjoint $\mathfrak{r}$ action on $\Lambda^*(\mathfrak{u})$ differs from the $\mathfrak{spin}(\mathfrak{s})$ action on $S = \Lambda^*(\overline{\mathfrak{u}}$ by a scalar factor $\mathbf{C}_{\rho(\mathfrak{u})}$. Here $\rho(\mathfrak{u})$ is half the sum of the weights in $\mathfrak{u}$.

The Dirac operator in this situation can be written (for details of this, see [12]) as the sum

$$\not{D}_{\mathfrak{g},\mathfrak{r}} = C^+ + C^-$$

where (using dual bases u_i for $\mathfrak{u}$ and u_i^* for $\mathfrak{u}^*$)

$$C^+ = \sum_i u_i^* \otimes u_i + 1 \otimes \frac{1}{4} \sum_{i,j} u_i u_j [u_i^*, u_j^*]$$

$$C^- = \sum_i u_i \otimes u_i^* + 1 \otimes \frac{1}{4} \sum_{i,j} u_i^* u_j^* [u_i, u_j]$$

The operators C^+ and C^- are differentials satisfying $(C^-)^2 = (C^+)^2 = 0$, and negative adjoints of each other. This is very much like the standard Hodge theory set-up, and one has

$$V \otimes S = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^+ \oplus Im\ C^-$$

$$Ker\ C^+ = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^+$$

$$Ker\ C^- = Ker\ \not{D}_{\mathfrak{g},\mathfrak{r}} \oplus Im\ C^-$$

If one identifies $S = \Lambda^*(\mathfrak{u})$, then $V \otimes S$ with differential C^+ is the complex with cohomology $H^*(\overline{\mathfrak{u}}, V)$, and if one identifies $S = \Lambda^*(\overline{\mathfrak{u}})$, then $V \otimes S$ with differential C^- is the complex with homology $H_*(\mathfrak{u}, V)$. Both of these can be identified with the cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V) = Ker\ D_{\mathfrak{g},\mathfrak{r}}$. Note that one gets not the usual $\mathfrak{r}$ action on $H^*(\overline{\mathfrak{u}}, V)$ or $H_*(\mathfrak{u}, V)$, but the action twisted by the one-dimensional representation $\mathbf{C}_{\rho(\mathfrak{u})}$.

6 Examples

For specific choices of $\mathfrak{g}$ and $\mathfrak{r}$ one can In each case, determination of the Dirac cohomology $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V)$ depends upon the formula

$$\not{D}^2 = -\Omega_{\mathfrak{g}} \otimes 1 + \zeta(\Omega_{\mathfrak{r}}) + (||\rho_{\mathfrak{r}}||^2 - ||\rho_{\mathfrak{g}}||^2) 1 \otimes 1$$

and the fact that $\Omega_{\mathfrak{g}}$ acts by the scalar

$$||\lambda + \rho||^2 - ||\rho||^2$$

on an irreducible representation of highest-weight λ.

6.1 The cases $\mathfrak{r} = 0$ and $\mathfrak{r} = \mathfrak{g}$

- $\mathfrak{r} = \emptyset$

  This is the case of no un-gauged symmetry, corresponding to the quantum Weil algebra itself. Here
  $$H_{\not{D}}(\mathfrak{g}, \emptyset) = \mathbf{C}$$
  and
  $$H_{\not{D}}(\mathfrak{g}, \emptyset; V) = \emptyset$$
  for any V.

- $\mathfrak{r} = \mathfrak{g}$

  This is the case of no gauged symmetry. Here one has
  $$H_{\not{D}}(\mathfrak{g}, \mathfrak{g}) = Z(\mathfrak{g})$$
  and
  $$H_{\not{D}}(\mathfrak{g}, \mathfrak{g}; V) = V$$

6.2 Generalized Highest-weight Theory

The case $\mathfrak{r} = \mathfrak{h}$, the Cartan subalgebra of a complex semi-simple Lie algebra reproduces the Cartan-Weyl highest-weight theory, of finite dimensional representations, as generalized by Bott[3] and Kostantkostant-borelweil. In this case
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \overline{\mathfrak{n}}$$
and one can identify Dirac cohomology and Lie algebra cohomology in the manner discussed above. One has
$$H_{\not{D}}(\mathfrak{g}, \mathfrak{h}) = Z(\mathfrak{h}) = S^*(\mathfrak{h}^*)$$
and
$$H_{\not{D}}(\mathfrak{g}, \mathfrak{h}; V_\lambda) = \sum_{w \in W} \mathbf{C}_{w(\lambda + \delta_{\mathfrak{g}})}$$
which has dimension $|W|$.

The Weyl-character formula for the character $ch(V_\lambda)$ function on the Cartan sub-algebra can be derived from this, by taking a supertrace, i.e. the difference between the part of the Dirac cohomology lying in the half spinor S^+ and that lying in the half spinor S^-. One has
$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W} (-1)^{l(w)} \mathbf{C}_{w(\lambda + \delta_{\mathfrak{g}})}$$
as $\mathfrak{h}$ represenions. So the character satisfies
$$ch(V_\lambda) = \frac{ch(V_\lambda \otimes S^+ - V_\lambda \otimes S^-)}{ch(S^+ - S^-)}$$

which is

$$\frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\delta_{\mathfrak{g}})}}{\sum_{w\in W}(-1)^{l(w)}e^{w(\delta_{\mathfrak{g}})}}$$

More generally, if $\mathfrak{p}=\mathfrak{l}\oplus\mathfrak{u}$ is a parabolic sub-algebra of $\mathfrak{g}$, with Levi factor $\mathfrak{l}$ and nilradical $\mathfrak{u}$, then again Dirac cohomology can be identified with the Lie algebra cohomology for the subalgebra $\mathfrak{u}$. One finds

$$H_{\not{D}}(\mathfrak{g},\mathfrak{l})=Z(\mathfrak{l})$$

and

$$H_{\not{D}}(\mathfrak{g},\mathfrak{l};V_\lambda)=\sum_{w\in W/W_{\mathfrak{l}}}V_{w(\lambda+\delta_{\mathfrak{g}})-\delta_{\mathfrak{l}}}$$

which is a sum of $|W/W_{\mathfrak{l}}|$ $\mathfrak{l}$-modules.

6.3 $(\mathfrak{g},K)$ modules

For real semi-simple Lie algebras $\mathfrak{g}_0$, corresponding to real Lie groups G with maximal compact subgroup K (with Lie algebra $\mathfrak{k}_0$), the interesting unitary representations are infinite-dimensional. The simplest example here is $\mathfrak{g}_0=\mathfrak{sl}(2,\mathbf{R})$, $K=SO(2)$ which has important applications in the theory of automorphic forms. These representations can be studied in terms of the corresponding Harish-Chandra $(\mathfrak{g},K)$ modules, using relative Lie algebra cohomology $H^*(\mathfrak{g},K;V)$ to produce invariants of the representations (see [13] and [14]). Here $\mathfrak{g},\mathfrak{k}$ are the complexifications of $\mathfrak{g}_0,\mathfrak{k}_0$. Dirac cohomology can also be used in this context, and one finds(see [12], chapter 8)

$$H^*(\mathfrak{g},K;V\otimes F^*)=Hom_{\mathfrak{k}}(H_{\not{D}}(\mathfrak{g},\mathfrak{k};F),H_{\not{D}}(\mathfrak{g},\mathfrak{k};V))$$

when V is an irreducible unitary $(\mathfrak{g},K)$ module with the same infinitesimal character as a finite dimensional $(\mathfrak{g},K)$ module F.

6.4 Examples not related to Lie algebra cohomology

A remarkable property of Dirac cohomology is that it exists and allows the definition of a physical space of states for a system with a symmetry group G, where a specified subgroup R remains ungauged, even in cases where $\mathfrak{g}/\mathfrak{r}$ is not a Lie algebra and is not even of the form $\mathfrak{u}\oplus\overline{\mathfrak{u}}$ for $\mathfrak{u}$ a Lie algebra. In other words, there is no Lie algebra to apply Lie algebra cohomology and the BRST method to.

A simple example is the case $\mathfrak{g}=\mathfrak{spin}(2n+1),\mathfrak{r}=\mathfrak{spin}(2n)$. Here for $n>1$ $\mathfrak{spin}(2n+1)/\mathfrak{spin}(2n)$ cannot be decomposed as $\mathfrak{u}\oplus\overline{\mathfrak{u}}$, which corresponds to the fact that even dimensional spheres S^{2n} cannot be given an invariant complex structure for $n>1$. In this case

$$H_{\not{D}}(\mathfrak{spin}(2n+1),\mathfrak{spin}(2n))=Z(\mathfrak{spin}(2n))$$

and

$$H_{\not{D}}(\mathfrak{spin}(2n+1), \mathfrak{spin}(2n); V_\lambda)$$

is a sum of two $\mathfrak{spin}(2n)$ representations. For the trivial representation, these are just the two half-spinor representations of $\mathfrak{spin}(2n)$.

7 Coupling the Kostant-Dirac Operator to a Connection

In [5] Freed, Hopkins and Teleman use a construction related to the one discussed in this paper. It involves studying not a single algebraic Dirac operator $\not{D}_\mathfrak{g}$, but a family of them parametrized by $\mathfrak{g}^*$. For $\mu \in g^*$, one takes the operator

$$\not{D}_{\mathfrak{g},\mu} = \not{D}_\mathfrak{g} + 1 \otimes \mu$$

interpreting μ as an element of $Cliff(\mathfrak{g})$. This is an algebraic version of the construction of coupling a Dirac operator to a connection, with connections corresponding to elements of $\mathfrak{g}^*$.

While $\not{D}_\mathfrak{g}$ acting on $V_\lambda \otimes S$ has no kernel, $\not{D}_{\mathfrak{g},\mu}$ will have a kernel when μ is in a specifi co-adjoint orbit. The Kirillov correspondence maps co-adjoint orbits to irreducible representations, in this case by the Borel-Weil theorem which shows how to produce an irreducible representation from a co-adjoint orbit. The FHT construction of a family of Dirac operators gives an inverse to this map, associating co-adjoint orbits to irreducible representations. This is somewhat analogous to the way in which Beilinson-Bernstein localization produces D-modules on flag varieties from representations.

FHT use this construction not for finite dimensional Lie algebras, but for affine Kac-Moody algebras, finding a theorem identifying the Verlinde ring of these algebras with twisted equivariant K-theory of the corresponding finite-dimensional group.

8 Gauge Theory in $0+1$ dimensions

The physical case for which one needs to handle quantum gauge symmetry is that of G-gauge theory (G a compact, connected Lie group) coupled to spinor fields twisted by a representation V of G in 3+1 dimensions. A toy model for this would be to take gauge theory in 0+1 dimensions, twisted by a representation V of G. Taking spacetime to be a circle S^1, the basic geometrical set-up is that of a principal G-bundle P over S^1. The fields consist of connections $A \in \Omega(P) \otimes \mathfrak{g}$. and sections of the associated bundle E_V with fiber V. The gauge group $\mathcal{G}$ is the group of vertical automorphisms of the bundle. Since the bundle has a section, $\mathcal{G}$ can be identified with $Maps(S^1 \to G)$.

The subgroup $\mathcal{G}_0 \subset \mathcal{G}$ of based gauge transformations acts freely on the space of connections, with quotient $\mathcal{A}/\mathcal{G}_0$ isomorphic to G, which can be identified with the group element one gets from parallel transport around the circle starting and ending at the base-point. The remaining gauge symmetry group is $\mathcal{G}/\mathcal{G}_0$ which is isomorphic to G and acts on the quotient $\mathcal{A}/\mathcal{G}_0$ by conjugation.

Alternatively, what we have done is used the gauge-freedom to make the connection time independent, so now our space of connections is just the constant $A_0 \in \mathfrak{g}$, with residual gauge symmetry adjoint action of G on $\mathfrak{g}$. This action is not free. One way to deal with this remaining gauge symmetry is to pick a maximal torus T (and thus a Cartan subalgebra $\mathfrak{t} = Lie(T)$ of $\mathfrak{g}$), and choose as gauge condition $A_0 \in \mathfrak{t}$. After this, the only remaining gauge symmetry will be the T symmetry. The T symmetry will act trivially on the space of connections, but it will act non trivially on states, since they include the field V.

So, in this case, we are in exactly the situation where Dirac cohomology can be applied, giving as space of physical states $H_{\not{D}}(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}; V)$. This is exactly one of the examples described earlier, the one that can be used to derive the Weyl character formula. The Hilbert space is $V \otimes S$, with physical states in $Ker\ \not{D}_{\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}}}$. Operators are in $U(\mathfrak{g}) \otimes Cliff(\mathfrak{n} \oplus \overline{\mathfrak{n}})$, with physical operators cohomology classes, isomorphic to $S(\mathfrak{t})$.

9 Higher Dimensional Applications

9.1 Geometric Langlands

BRST symmetry has previously found a role in the geometric Langlands program, and it may be possible to use Dirac cohomology for similar purposes. In local geometric Langlands, the relevant quantum field theory is a QFT on a disc, with Hilbert space a representation of a loop grou. On hopes to parametrize reps by an analog of langlands parameters, which here can be thought of as connections with Langlands dual group.

Feigin-Frenkel implement this by showing that the center of the enveloping algebra for the affine Lie algebra at the critical level is the space of functions on the Drinfeld-Sokolov reduction of the right space of connections. They relate this to a QFT, gauged WZNW, gauging the complex group WZW model by the nilpotent subgroup.

It may be possible to instead use the Dirac cohomology version of the center, ending up with a different QFT to study. A construction of this kind may have interesting generalizations to higher dimensions, involving QFTs much like the standard model, with connections parametrizing higher dimensional gauge group representations.

9.2 Geometric Dirac Operators

The Dirac operator appearing here is an algebraic version of the Dirac operator. For the case of a group manifold, one can get geometric Dirac operators by taking the representation to be functions on the group.

Relation to the supersymmetric quantum mechanics proof of the index theorem, which has sometimes been claimed to come from a BRST-fixing of infinitesimal translations on the manifold.

9.3 Quantization of Higher-Dimensional Gauge Theory

The formalism discussed in this paper has been purely a Hamiltonian formalism. To generalize to higher dimensions, one would like a Lagrangian formalism, and to make contact with the path integral and the Faddev-Popov treatment of gauge fixing. This might resolve some of the problems that have so far thwarted attempts to treat BRST symmetry non-perturbatively.

10 Conclusion

In a simplified, finite-dimensional context, we have shown that Dirac cohomology gives an analog of BRST method for treating gauge symmetry, where

- $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}; V)$ are the analog of BRST-invariant states
- $H_{\not{D}}(\mathfrak{g}, \mathfrak{r}) = Z(\mathfrak{r})$ are the analog of BRST-invariant operators, acting on these states.

The operator $\not{D}_{\mathfrak{g},\mathfrak{r}}$ plays a role analogous to that of the BRST operator Q, but its square is not zero.

In the case $\mathfrak{s} = \mathfrak{u} \oplus \overline{\mathfrak{u}}$, this is identical to the BRST formalism, up to a twist by a scalar factor. It applies also in more general cases where there is Lie algebra like $\mathfrak{u}$ to use for the BRST formalism.

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