**Groups and Representations II: Problem Set 1**

**Due Monday, February 5**

**Problem 1:** If $V$ is a finite-dim representation of a finite group $G$, show that the set of $G$ invariant inner products on $V$ is isomorphic to $\text{Hom}_G(V^*, V^*)$. Use this to conclude that, up to scalar multiplication, there is a unique $G$-invariant inner product on an irreducible representation $V$. (Note: this is exercise 2.22 in Sepanski)

**Problem 2:** Given two finite groups $G_1$ and $G_2$, show that all irreducible representations of $G_1 \times G_2$ are of the form $V_1 \otimes V_2$, where $V_1$ is an irreducible representation of $G_1$ and $V_2$ is an irreducible representation of $G_2$.

**Problem 3:** Work out the details of the proof outlined in class showing that the right action of $G$ on itself induces a representation on $\text{Hom}_G(V_i, C(G))$ isomorphic to $V_i^*$. Hint: Sepanski works this out in the more general compact group case on pages 62 and 63.

**Problem 4:** For the the group $S^3$, work out:

- The explicit form of the three irreducible representations (i.e. find the matrices in a chosen basis).
- The characters of the irreducible representations.
- The decomposition of the tensor product $V \otimes V$ into irreducibles, where $V$ is the 2-dimensional irreducible.

**Problem 5:** Show that the matrix elements of irreducible representations of a finite group are orthogonal (as functions on the group).

**Problem 6:** Given a finite-dimensional representation $V$, one can form representations $S^2 V$ and $\Lambda^2 V$ by taking the symmetric and antisymmetric parts of the tensor product $V \otimes V$. Show that

$$\chi_{S^2 V} = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$$

and

$$\chi_{\Lambda^2 V} = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

and thus

$$V \otimes V = S^2 V \oplus \Lambda^2 V$$