Groups and Representations II: Problem Set 1

Due Monday, February 5

Problem 1: If V is a finite-dim representation of a finite group G, show that the set of G invariant inner products on V is isomorphic to $Hom_G(V^*, V^*)$. Use this to conclude that, up to scalar multiplication, there is a unique G-invariant inner product on an irreducible representation V. (Note: this is exercise 2.22 in Sepanski)

Problem 2: Given two finite groups G_1 and G_2, show that all irreducible representations of $G_1 \times G_2$ are of the form $V_1 \otimes V_2$, where V_1 is an irreducible representation of G_1 and V_2 is an irreducible representation of G_2.

Problem 3: Work out the details of the proof outlined in class showing that the right action of G on itself induces a representation on $Hom_G(V_i, \mathbf{C}(G))$ isomorphic to V_i^*. Hint: Sepanski works this out in the more general compact group case on pages 62 and 63.

Problem 4: For the the group S^3, work out:

- The explicit form of the three irreducible representations (i.e. find the matrices in a chosen basis).
- The characters of the irreducible representations.
- The decomposition of the tensor product $V \otimes V$ into irreducibles, where V is the 2-dimensional irreducible.

Problem 5: Show that the matrix elements of irreducible representations of a finite group are orthogonal (as functions on the group).

Problem 6: Given a finite-dimensional representation V, one can form representations S^2V and $\Lambda^2 V$ by taking the symmetric and antisymmetric parts of the tensor product $V \otimes V$. Show that

$$\chi_{S^2V} = \frac{1}{2}(\chi_V(g)^2 + \chi_V(g^2))$$

and

$$\chi_{\Lambda^2V} = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))$$

and thus

$$V \otimes V = S^2V \oplus \Lambda^2 V$$