Problem 1: Consider the group $SU(1,1)$, defined as the group of two by two complex matrices $g$ of determinant one, satisfying

$$g^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Show that

$$SU(1,1) = \{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \}$$

- Show that, if

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

then

$$g \in SU(1,1) \leftrightarrow TgT^{-1} \in SL(2,\mathbb{R})$$

and that this gives an isomorphism of the groups $SL(2,\mathbb{R})$ and $SU(1,1)$.

- Under this isomorphism, what element of $SU(1,1)$ corresponds to the standard complex structure $J_0$ in $SL(2,\mathbb{R})$?

- Which subgroups of $SL(2,\mathbb{R})$ and $SU(1,1)$ commute with the standard complex structure? In other words, give explicitly the group of matrices $g$ such that $gJ_0 = J_0g$ in both cases.

Problem 2: Show that the linear transformation of annihilation and creation operators

$$a \to a' = \alpha a + \beta a^\dagger$$

$$a^\dagger \to (a')^\dagger = \bar{\beta}a + \bar{\alpha}a^\dagger$$

preserves the commutation relations iff

$$|\alpha|^2 - |\beta|^2 = 1$$

Such transformations are known as “Bogoliubov transformations”. For which Bogoliubov transformations does $a|0\rangle = 0$ imply $a'|0\rangle = 0$?

Problem 3: For the coherent state $|\alpha\rangle$, compute

$$\langle \alpha | Q | \alpha \rangle$$
and

\[ \langle \alpha | P | \alpha \rangle \]

Show that coherent states are not eigenstates of the number operator \( N = a^\dagger a \) and compute

\[ \langle \alpha | N | \alpha \rangle \]

Similarly, for squeezed states, compute

\[ \tau \langle 0 | N | 0 \rangle \tau \]

as a function of \( \tau \).

**Problem 4:** Find the coherent state \( |\alpha\rangle \) in the Schrödinger representation. In other words, find a wavefunction \( \psi(q) \) that satisfies

\[
\frac{1}{\sqrt{2}}(Q + iP)\psi(q) = \alpha \psi(q)
\]