Lie Groups and Representations, Spring 2016
Problem Set 3

Due Monday, February 15

Problem 1:

- If $R \subset E$ is a root system, let $R^\vee \subset E^*$ be the set of elements given by the co-roots $\alpha^\vee$ for $\alpha \in R$. Show that $R^\vee$ is a root system.
- Show that if $R = A_n$, then $R^\vee = A_n$, and that if $R = B_n$, then $R^\vee = C_n$.

Problem 2:

- For the standard basis E, F, H of $\mathfrak{sl}(2, \mathbf{C})$, show that

$$S = exp(\frac{\pi}{2}(E - F)) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  and that

$$Ad_S(H) = -H$$

- Assume that the Lie algebra $\mathfrak{g}$ is given as a sub-Lie algebra of $\mathfrak{gl}(n, \mathbf{C})$, with compact real form $\mathfrak{k}$ and $K \subset GL(n, \mathbf{C})$ the compact subgroup with Lie algebra $\mathfrak{k}$. For each root α of $\mathfrak{g}$, construct $\mathfrak{sl}(2, \mathbf{C})_\alpha$ as in class, and use the above construction to construct a group element $S_\alpha \subset SU(2) \subset K$.

  Show that Ad_{S_α} satisfies

$$Ad_{S_\alpha}(\alpha^\vee) = -\alpha^\vee$$

  and

$$Ad_{S_\alpha}(H) = H$$

  if $H \in \mathfrak{h}$ such that $\alpha(H) = 0$. Deduce from this that S_α acts on $\mathfrak{g}^*$ preserving $\mathfrak{h}^*$, in such a way that its restriction to $\mathfrak{h}^*$ is the Weyl reflection s_α.

Problem 3: Let

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$$

Show that

$$\rho(\alpha_i^\vee) = 1$$

for each co-root of a simple root.

Problem 4: For the root system G_2, start with the simple roots, identify the Weyl group, drawing the Weyl chambers and full root system in the plane.