Problem 1:

- If $R \subset E$ is a root system, let $R^\vee \subset E^*$ be the set of elements given by the co-roots $\alpha^\vee$ for $\alpha \in R$. Show that $R^\vee$ is a root system.
- Show that if $R = A_n$, then $R^\vee = A_n$, and that if $R = B_n$, then $R^\vee = C_n$.

Problem 2:

- For the standard basis $E, F, H$ of $\mathfrak{sl}(2, \mathbb{C})$, show that
  $$ S = \exp\left(\pi \frac{1}{2} (E - F)\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} $$
  and that
  $$ \text{Ad}_S(H) = -H $$
- Assume that the Lie algebra $\mathfrak{g}$ is given as a sub-Lie algebra of $\mathfrak{gl}(n, \mathbb{C})$, with compact real form $\mathfrak{k}$ and $K \subset GL(n, \mathbb{C})$ the compact subgroup with Lie algebra $\mathfrak{k}$. For each root $\alpha$ of $\mathfrak{g}$, construct $\mathfrak{sl}(2, \mathbb{C})_\alpha$ as in class, and use the above construction to construct a group element $S_\alpha \subset SU(2) \subset K$.
  Show that $\text{Ad}_{S_\alpha}$ satisfies
  $$ \text{Ad}_{S_\alpha}(\alpha^\vee) = -\alpha^\vee $$
  and
  $$ \text{Ad}_{S_\alpha}(H) = H $$
  if $H \in \mathfrak{h}$ such that $\alpha(H) = 0$. Deduce from this that $S_\alpha$ acts on $\mathfrak{g}^*$ preserving $\mathfrak{h}^*$, in such a way that its restriction to $\mathfrak{h}^*$ is the Weyl reflection $s_\alpha$.

Problem 3: Let
  $$ \rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha $$
  Show that
  $$ \rho(\alpha_i^\vee) = 1 $$
  for each co-root of a simple root.

Problem 4: For the root system $G_2$, start with the simple roots, identify the Weyl group, drawing the Weyl chambers and full root system in the plane.