Problem 1: For the Killing form $K(X, Y) = \text{tr}(\text{ad}X \circ \text{ad}Y)$

- For the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$, this is a bilinear form on a 3-dimensional space. With respect to the standard basis, write this as a 3 by 3 matrix.

- Show that the Killing form on the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ is given by
  
  $$K(X, Y) = 2n \text{tr}(XY) - 2\text{tr}(X)\text{tr}(Y)$$

  Show that this is non-degenerate only on the subalgebra $\mathfrak{sl}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$.

Problem 2: Find a Cartan subalgebra for $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C})$. What are the roots $\alpha \in \mathbb{R}$ and root spaces $\mathfrak{g}_\alpha$?

Problem 3: For $\mathfrak{g}$ a complex simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and set of roots $R$. For each root $\alpha \in R$, show the the construction of the Lie subalgebra $\mathfrak{sl}(2, \mathbb{C})_\alpha$ given in class is unique up to

- rescaling by a complex constant $c$

  $$E_\alpha \rightarrow cE_\alpha, \quad F_\alpha \rightarrow c^{-1}F_\alpha, \quad \alpha^\vee \rightarrow c\alpha^\vee$$

- making the change

  $$E_\alpha \leftrightarrow F_\alpha, \quad \alpha^\vee \rightarrow -\alpha^\vee$$