Lie Groups and Representations, Spring 2016
Problem Set 1

Due Monday, February 1

Problem 1: For the Killing form $K(X, Y) = tr(adX \circ adY)$

- For the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, this is a bilinear form on a 3-dimensional space. With respect to the standard basis, write this as a 3 by 3 matrix.
- Show that the Killing form on the Lie algebra $\mathfrak{gl}(n, \mathbf{C})$ is given by

$$K(X, Y) = 2n\ tr(XY) - 2tr(X)tr(Y)$$

Show that this is non-degenerate only on the subalgebra $\mathfrak{sl}(n, \mathbf{C}) \subset \mathfrak{gl}(n, \mathbf{C})$.

Problem 2: Find a Cartan subalgebra for $\mathfrak{g} = \mathfrak{so}(4, \mathbf{C})$. What are the roots $\alpha \in R$ and root spaces $\mathfrak{g}_\alpha$?

Problem 3: For $\mathfrak{g}$ a complex simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$ and set of roots R. For each root $\alpha \in R$, show the the construction of the Lie subalgebra $\mathfrak{sl}(2, \mathbf{C})_\alpha$ given in class is unique up to

- rescaling by a complex constant c

$$E_\alpha \to cE_\alpha, \quad F_\alpha \to c^{-1}F_\alpha, \quad \alpha^\vee \to \alpha^\vee$$

- making the change

$$E_\alpha \leftrightarrow F_\alpha, \quad \alpha^\vee \to -\alpha^\vee$$