The Weyl Character Formula

Math G4344, Spring 2012

1 Characters

We have seen that irreducible representations of a compact Lie group G can be constructed starting from a highest weight space and applying negative roots to a highest weight vector. One crucial thing that this construction does not easily tell us is what the character of this irreducible representation will be. The character would tell us not just which weights occur in the representation, but with what multiplicities they occur (this multiplicity is one for the highest weight, but in general can be larger). Knowing the characters of the irreducibles, we can use this to compute the decomposition of an arbitrary representation into irreducibles.

The character of a representation (π, V) is the complex-valued, conjugation-invariant function on G given by

$$\chi_V(g) = Tr(\pi(g))$$

The representation ring $R(G)$ of a compact Lie group behaves much the the same as in the finite group case, with the sum over group elements replaced by an integral in the formula for the inner product on $R(G)$

$$< [V], [W] >= \int_G \overline{\chi_V(g)}\chi_W(g)dg$$

where dg is the invariant Haar measure giving G volume 1. As an inner product space $R(G)$ has a distinguished orthonormal basis given by the characters $\chi_{V_i}(g)$ of the irreducible representations. For an arbitrary representation V, once we know its character χ_V we can compute the multiplicities n_i of the irreducibles in the decomposition

$$V = \bigoplus_{i\in\hat{G}} n_i V_i$$

as

$$n_i = \int \overline{\chi_{V_i}(g)}\chi_V(g)dg$$

Knowing the character of V is equivalent to knowing the weights γ_i in V, together with their multiplicities n_i, since if

$$V = \bigoplus_i n_i \mathbf{C}_{\gamma_i}$$

then

$$Tr_V(e^H) = \sum_i n_i e^{\gamma_i(H)}$$

2 The Weyl Integral Formula

We'll give a detailed outline of Weyl's proof of the character formula for a highest-weight irreducible. This exploits heavily the orthonormality properties of characters, and requires understanding the relation between the inner products on $R(G)$ and $R(T)$, which is described by an integral formula due to Weyl.

We have seen that representations of G can be analyzed by considering their restrictions to a maximal torus T. On restricting to T, the character $\chi_V(g)$ of a representation V gives an element of $R(T)$ which is invariant under the Weyl group $W(G,T)$. We need to be able to compare the inner product $< \cdot, \cdot >_G$ on $R(G)$ to $< \cdot, \cdot >_T$, the one on $R(T)$. This involves finding a formula that will allow us to compute integrals of conjugation invariant functions on G in terms of integral over T, this will be the Weyl integral formula.

We'll be considering the map:

$$q : (gT, t) \in G/T \times T \to gtg^{-1} \in G$$

An element n of the Weyl group $W(G,T)$ (i.e. an element nT in $N(T)/T$, $N(T)$ the normalizer of T) acts on $G/T \times T$ by

$$(gT, t) \to gn^{-1}T, ntn^{-1}$$

and all points in a Weyl group orbit are mapped by q to the same point in G. On "regular" points away from a positive codimension locus of "singular" points of T the map q is a covering map with fiber $W(G,T)$. One way to see this is to consider an element

$$x \in \mathfrak{t} \subset \mathfrak{g}$$

Under the adjoint action

$$x \to gxg^{-1}$$

T leaves x invariant, elements of G/T move it around in an orbit in $\mathfrak{g}$, one that intersects T at points corresponding to the elements of the Weyl group. For x regular these points are all distinct (one in the interior of each Weyl chamber).

The map q is a covering map of degree $|W|$ away from the singular elements of T, these are of codimension one in T. As long as a function f on G doesn't behave badly at the images of the singular elements, one can integrate a function over G by doing the integral over the pull-back function $q^*(f)$ on $G/T \times T$, getting $|W|$ copies of the answer one wants.

So

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T \int_{G/T} q^*f(gT,t)q^*(dg)$$

One needs to compute the differential dq of the map q to get the Jacobian factor in

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T \int_{G/T} f(gtg^-1)|\det(dq)|d(gT)dt$$

The result of this calculation is the

Theorem 1 (Weyl Integral Formula)**.** *For f a continuous function on a compact connected Lie group G with maximal torus T one has*

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T det((\mathbf{1} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})(\int_{G/T} f(gtg^{-1})d(gT))dt$$

and if f is a conjugation-invariant function ($f(gtg^{-1}) = f(t)$), then

$$\int_G f(g)dg = \frac{1}{|W(G,T)|}\int_T det((\mathbf{1} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})f(t)dt$$

For a detailed proof of this formula, see [1] Chapter IV.1. The essence of the computation is computing the Jacobian of the map q. Using left invariance of the measures involved and translating the calculation to the origin of $G/T \times T$, in the decomposition $\mathfrak{g} = \mathfrak{g}/\mathfrak{t} + \mathfrak{t}$ the Jacobian is trivial on $\mathfrak{t}$, but picks up the non-trivial factor

$$J(t) = \det((\mathbf{I} - Ad(t^{-1}))_{|\mathfrak{g}/\mathfrak{t}})$$

from the the $\mathfrak{g}/\mathfrak{t}$ piece.

One can explicitly calculate the Jacobian factor $J(t)$ in terms of the roots, since they are the eigenvalues of Ad on $\mathfrak{g}/\mathfrak{t}$ with the result

$$J(t = e^H) = \prod_{\alpha \in R}(1 - e^{-\alpha(H)}) = \prod_{\alpha \in R^+}(1 - e^{\alpha(H)})(1 - e^{-\alpha(H)})$$

where R is the set of roots, R^+ is the set of positive roots. From this formula we can see that we can in some sense take the square root of $J(t)$, one choice of phase for this is to set

$$\delta(e^H) = e^{\rho}(H)\prod_{\alpha \in R^+}(1 - e^{-\alpha(H)}) = \prod_{\alpha \in R^+}(e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)})$$

where ρ is half the sum of the positive roots. Then

$$J(t) = \delta\bar{\delta}$$

So the final result is that for conjugation invariant f one has

$$\int_G f dg = \frac{1}{|W|}\int_T f(t)\delta(t)\overline{\delta(t)}dt$$

For the case of $G = U(n)$, the Weyl integration formula gives a very explicit formula for how to integrate a function of unitary matrices that only depends on their eigenvalues over the space of all unitary matrices. In this case an element of T is a set of n angles $e^{i\theta_i}$ and the Jacobian factor in this case is

$$\begin{aligned} J(\theta_1, \cdots, \theta_n) &= \prod_{\alpha \in R}(1 - e^{-\alpha}) \\ &= \prod_{j \neq k}(1 - e^{i(\theta_k - \theta_j)}) \\ &= \prod_{j<k}(1 - e^{i(\theta_j - \theta_k)})(e^{i\theta_k} e^{-i\theta_k})(1 - e^{i(\theta_k - \theta_j)}) \\ &= \prod_{j<k}(e^{i\theta_k} - e^{i\theta_j})(e^{-i\theta_k} - e^{-i\theta_j}) \\ &= \prod_{j<k}|e^{i\theta_k} - e^{i\theta_j}|^2 \end{aligned}$$

Note that this factor suppresses contributions to the integral when two eigenvalues become identical. The full integration formula becomes

$$\int_{U(n)} f = \frac{1}{n!}\int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \cdots, \theta_2) \prod_{j<k}|e^{i\theta_k} - e^{i\theta_j}|^2 \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi}$$

For the case $G = SU(2),\ T = U(1)$, one has

$$\begin{aligned} \int_{SU(2)} f(g)dg &= \frac{1}{2}\int_0^{2\pi} f(\theta)|e^{i\theta} - e^{-i\theta}|^2 \frac{d\theta}{2\pi} \\ &= 2\int_0^{2\pi} f(\theta)\sin^2(\theta)\frac{d\theta}{2\pi} \end{aligned}$$

3 The Weyl Character Formula

The Weyl integral formula was used by Weyl to derive a general formula for character of an irreducible G representation. In this section we'll give this character formula and outline its proof. This proof is somewhat indirect and we'll describe later some other different proofs.

Weyl's derivation of the character formula uses a somewhat surprising trick. A character of a G representation gives an element of $R(T)$ (explicitly a complex-valued function on T), one that is invariant under the Weyl group $W(G,T)$. The trick is to carefully analyze not the Weyl-symmetric character functions on T, but the antisymmetric characters, those elements of $R(T)$ that are antisymmetric under Weyl reflections. These are functions on T that change sign when one does a Weyl reflection in the hyperplane perpendicular to a simple root, i.e.

$$f(wt) = sgn(w)f(t)$$

where $sgn(w)$ is $+1$ for elements of $w \in W(G,T)$ built out of an even number of simple reflections, -1 for an odd number. It could also be written

$$sgn(w) = (-1)^{l(w)}$$

where $l(w)$ is the length of w. The character function we want will turn out to be the ratio of two such antisymmetric characters.

Theorem 2 (Weyl Character Formula)**.** *The character of the irreducible G representation V^λ with highest weight $\lambda \in \mathfrak{t}^*$ is*

$$\chi_{V^\lambda}(e^H) = \frac{\sum_{w\in W} sgn(w)e^{(\lambda+\rho)(wH)}}{\delta}$$

where $H \in \mathfrak{t}$, ρ is one-half the sum of the positive roots and δ is the function on T defined in the discussion of the Weyl integral formula

Outline of Proof:

Weyl's proof begins by showing that the antisymmetric characters

$$A_\gamma(e^H) = \sum_{w\in W} sgn(w)e^{\gamma(wH)}$$

for γ a weight in the dominant Weyl chamber form an integral basis for the vector space of all antisymmetric characters. Furthermore they satisfy the orthogonality relations

$$< A_{\gamma_i}, A_{\gamma_j} >_T = \int_T \overline{A}_{\gamma_i} A_{\gamma_j} = |W|\delta_{ij}$$

The irreducibility of V implies that $< \chi_V, \chi_V >_G = \int_G \overline{\chi_V}\chi_V = 1$ and the Weyl integral formula implies this is

$$1 = \frac{1}{|W|}\int_T \overline{\chi_V}\chi_V\overline{\delta}\delta dt = \frac{1}{|W|}\int_T \overline{(\chi_V\delta)}(\chi_V\delta)dt$$

Now $\chi_V\delta$ is an antisymmetric character since one can show that δ is antisymmetric under Weyl reflections and thus must be of the form

$$\chi_V\delta = \sum_\beta n_\beta A_\beta$$

where β are weights in the dominant Weyl chamber and n_β are integers. So we must have

$$\frac{1}{|W|}\int_T \overline{(\sum_\beta n_\beta A_\beta)}(\sum_\gamma n_\gamma A_\gamma) = 1$$

but by the orthogonality relations this is

$$1 = \sum_\beta (n_\beta)^2$$

so only one value of β can contribute, with coefficient ± 1. Finally one can show that for an irreducible representation with highest weight λ, $A_{\lambda+\delta}$ must occur in $\chi\delta$ with coefficient 1.

This finishes the outline of the proof, the details are in many textbooks, see for instance [1] Chapter VI.1.

As usual, the simplest example is the The Weyl character formula has a wide range of corollaries and applications, for example

Corollary 1 (Weyl Denominator Formula, alternate form of WCF)**.** *Applying the Weyl character formula to the case of the trivial representation gives*

$$\delta(e^H) = \sum_{w \in W} sgn(w) e^{\rho(wH)}$$

i.e.

$$\prod_{\alpha \in R^+} (e^{\frac{1}{2}\alpha(H)} - e^{-\frac{1}{2}\alpha(H)}) = \sum_{w \in W} sgn(w) e^{\rho(wH)}$$

This gives an alternate form of the Weyl character formula:

$$\chi_{V^\lambda}(e^H) = \frac{\sum_{w \in W} sgn(w) e^{(\lambda+\rho)(wH)}}{\sum_{w \in W} sgn(w) e^{\rho(wH)}}$$

As usual the simplest example to check is $G = SU(2),\ T = U(1)$, where the irreducible representations V^n of dimensions $n+1$ are labeled by $n = 0, 1, 2, \cdots$. From knowing that V^n has every other weight from $-n$ to $+n$, we know that

$$\chi_{V^n}(e^{i\theta}) = e^{-in\theta} + e^{-i(n-2)\theta} \cdots + e^{i(n-2)\theta} + e^{in\theta}$$

and multiplying this by $(e^{-i\theta} - e^{i\theta})$ we get a sum that collapses except for the first and last terms, giving exactly what the Weyl character formula predicts

$$\chi_{V^n}(e^{i\theta}) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

Using the Weyl integral formula, one can easily see that these are orthonormal

$$\int_{SU(2)} \overline{\chi_{V^n}} \chi_{V^m} dg = 2 \int_0^{2\pi} \frac{\sin((n+1)\theta)}{\sin(\theta)} \frac{\sin((m+1)\theta)}{\sin(\theta)} \sin^2(\theta) \frac{d\theta}{2\pi} = \delta_{nm}$$

Two other corollaries of the Weyl character formula are the following multiplicity formulae that you are asked to prove in this week's homework. The first is equivalent to the WCF, and some proofs of the WCF begin by proving it instead. Both use the Kostant partition function $P(\mu)$. which is defined as the number of ways one can write the integral weight μ as an integral combination of the positive roots.

Corollary 2 (Kostant Multiplicity Formula)**.** *An integral weight μ occurs in the highest weight representation V^λ with multiplicity*

$$\sum_{w \in W} (-1)^{l(w)} P(w(\lambda + \rho) - (\mu + \rho))$$

and

Corollary 3 (Steinberg Multiplicity Formula)**.** *The multiplicity of V^λ in the decomposition of the tensor product $V^\mu \otimes V^\nu$ is*

$$\sum_{w, w' \in W} (-1)^{l(w)l(w')} P(w(\lambda + \rho) + w'(\mu + \rho) - \nu - 2\rho)$$

4 The Weyl Dimension Formula

By evaluating the character formula at $H = 0$ one can compute the dimension of the irreducible representations giving

Corollary 4 (Weyl Dimension Formula)**.** *The dimension of the irreducible representation with highest weight λ is*

$$dim\ V_\lambda = \frac{\prod_{\alpha \in R^+} < \alpha, \lambda + \rho >}{\prod_{\alpha \in R^+} < \alpha, \rho >}$$

Here R^+ is the set of positive roots.

Proof:

This calculation is a little bit tricky since the Weyl character formula gives the character at $H = 0$ as the quotient of two functions that vanish to several orders there and so one has to use l'Hôpital's rule. We'll compute the values of the numerator and denominator of the Weyl character formula at $s\check{\rho} \in \mathfrak{t}$ (where $\vee$ denotes the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ given by the inner product, i.e. $\gamma(\check{\rho}) =< \gamma, \rho >$ for any $\gamma \in \mathfrak{t}^*$) and then take the limit as s goes to 0.

For the numerator of the Weyl character formula we have:

$$\begin{aligned}
A_{\lambda+\rho}(s\rho^\vee) &= \sum_{w \in W} sgn(w) e^{s<\lambda+\rho, w\rho>} \\
&= \sum_{w \in W} sgn(w) e^{s<w(\lambda+\rho), \rho>} \\
&= A_\rho(s(\lambda+\rho)^\vee) \\
&= \delta(s(\lambda+\rho)) \\
&= \prod_{\alpha \in R^+} (e^{\frac{s}{2}<\alpha,\lambda+\rho>} - e^{-\frac{s}{2}<\alpha,\lambda+\rho>}) \\
&= \prod_{\alpha \in R^+} s < \alpha, \lambda+\rho > + O(s^{n+1})
\end{aligned}$$

where n is the number of positive roots. Taking the ratio of this to the same expression with $\lambda = 0$ and taking the limit as s goes to zero gives the corollary.

In the $SU(2)$ case there is just one root and the formula gives $\frac{n+1}{1}$ as expected.

If an irreducible representation with highest weight ρ exists, the Weyl dimension formula implies that its dimension will be

$$\dim V_\rho = 2^m$$

where m is the number of positive roots. For $G = SU(3)$, this is the adjoint representation, which has dimension $8 = 2^3$ as predicted. We will see this representation occurring again when we study spinors.

5 Other proofs

We've outlined in detail Weyl's original proof of the Weyl character formula, but there are several other interesting well-known ones, invoking rather different kinds of mathematics, always in a quite non-trivial way. Some examples are:

- One can prove the WCF purely algebraically using Verma modules, beginning by calculating the character of the Verma module itself, which is quite easy:

  $$\chi_{V(\lambda)} = \frac{e^{\lambda}}{\prod_{\alpha \in R^+}(1 - e^{-\alpha})}$$

  The Verma module is infinite dimensional, so this formula should be interpreted as a formal sum, using

  $$\frac{1}{1 - e^{-\alpha}} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots$$

  If you take as given the BGG resolution of a finite-dimensional representation V^{λ} with highest-weight λ in terms of Verma modules, the WCF follows straightforwardly (showing this is one problem in this week's problem set).

  The WCF can also be proved using Verma modules and weaker information than the BGG resolution. You can use the Harish-Chandra isomorphism, which we will discuss next time (this proof is given in detail in [3]). It can also be done using even weaker information about the center $Z(\mathfrak{g})$, just about the Casimir, see for example [2].

- One can also prove the WCF using the construction of V^{λ} as holomorphic sections of a line bundle that one gets from the Borel-Weil theorem. Here one shows that for λ integral and dominant $V^{\lambda} = H^0(G/T, L_{\lambda})$ and that the higher cohomology $H^i(G/T, L_{\lambda}),\ i > 0$ vanishes. For the alternating sum of these cohomology groups the Atiyah-Bott-Lefschetz formula (a generalization of the Lefschetz trace formula to holomorphic complexes) gives a formula for the character evaluated at a generic element of the torus in terms of contributions from the fixed points of the torus action on G/T. There are $|W|$ of these, and the fixed-point contributions are exactly the terms in the WCF. For more details, see [4] or [5].

- A third and in many ways most straighforward proof of the WCF can be given once one knows the Lie algebra cohomology groups $H^i(\mathfrak{n}^+, V^{\lambda})$. These turn out to also govern what happens when one takes line bundles corresponding to non-dominant weights in the Borel-Weil construction. One gets representations in higher cohomology, and a generalization of the Borel-Weil theorem known as the Borel-Weil-Bott theorem. We'll cover this material in some detail a bit later in the course.

References

[1] Brocker, T. and tom Dieck, T., *Representations of Compact Lie Groups,* Springer-Verlag, 1985.

[2] Fulton, W., Harris, J., *Representation Theory: A First Course,* Springer-Verlag, 1991 Chapter 25.2.

[3] Knapp, A. *Lie Groups Beyond an Introduction, Second Edition* Birkhauser, 2002. Chapter V.6

[4] Pressley, A., and Segal, G., *Loop Groups,* Oxford University Press, 1986. Chapter 14.2.

[5] Bott, R., On Induced Representations, in *The Mathematical Heritage of Hermann Weyl, Proceedings of Symposia in Pure Mathematics, vol.48,* American Mathematical Society, 1988.