Generalities about Representation Theory

Math G4344, Spring 2012

I'll review here some basic facts about representations. Much of this should be familiar from last semester.

1 Representations, a review

For groups in general

Definition 1 (Group Representation). *A representation (π, V) of a group G on a vector space V is a homomorphism*

$$\pi : G \rightarrow GL(V)$$

where $GL(V)$ is the group of invertible linear transformations of V.

and for Lie algebras

Definition 2 (Lie algebra Representation). *A representation of a Lie algebra $\mathfrak{g}$ is a module for the algebra $U(\mathfrak{g})$. This module is given by a choice of vector space V, and a homomorphism $\pi : U(\mathfrak{g}) \rightarrow End(V)$.*

In the group case, one can alternatively define a representation as a module for the group algebra $\mathbf{C}G$ (this is an algebra of functions, with product given by convolution), we may say more about this when we discuss the Peter-Weyl theorem.

We'll sometime refer to a representation (π, V) as V, with the action of G or $\mathfrak{g}$ implicit, sometimes refer to it just using the homomorphism π.

Note that these definitions make sense for vector spaces (respectively modules) over an arbitrary field k. We'll restrict our attention to the simplest case, $k = \mathbf{C}$. The groups and Lie algebras themselves are also defined over a field, which for us will be $\mathbf{R}$ or $\mathbf{C}$. More generally, for the groups we'll consider one can keep track of whether the complex representations we study are "real" or "quaternionic". If a G-equivariant anti-linear map $J : V \rightarrow V$ exists satisfying $J^2 = 1$, one can think of J as a complex conjugation and restriction to its $+1$ eigenspace gives a representation on a real vector space. If a G-equivariant anti-linear map $J : V \rightarrow V$ exists satisfying $J^2 = -1$, one can use this J to give V the structure of a vector space over the quaternions.

For applications of representation theory in number theory, one may want to consider representations over the l-adic numbers $\mathbf{Q}_l$ for l a prime.

To study representations, one would like to decompose them into simpler ones,

Definition 3 (Irreducible Representation). *A representations is irreducible if it has no non-trivial sub-representations (i.e. no non-trivial subspace of V is invariant under the group action).*

A weaker property is

Definition 4 (Indecomposable Representation). *A representation is indecomposable if it is not the direct sum of two non-trivial proper subrepresentations.*

We'll see that for semi-simple Lie groups irreducibility and indecomposability coincide, but for solvable Lie groups one can have representations that are indecomposable, but not irreducible. For solvable Lie groups one has

Theorem 1 (Lie-Kolchin theorem). *If (π, V) is a representation of a solvable Lie group G of dimension n, there is a complete flag*

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V$$

of invariant subspaces V_i of dimension i.

This implies that any such representation can be put in the form of a subgroup of the upper-triangular matrices pf $GL(V)$, and any solvable Lie group has a one-dimensional irreducible representation (V_1). One may however not be able to find a G-invariant complement to V_1, for example in the case of all upper-triangular matrices in $GL(V)$ acting on V.

For many purposes it is a good idea to think of the set of representations of G not just as a set, but as a category $Rep(G)$. We can take the objects of this category to be the equivalence classes of representations,

Definition 5 (Equivalence of Representations). *Two representations π_1 and π_2 on a vector space V are said to be equivalent (or isomorphic) if they are related by conjugation, i.e.*

$$\pi_2(g) = h(\pi_1(g))h^{-1}$$

for $h \in GL(V)$.

The morphisms in the category $Rep(G)$ are not arbitrary linear maps between the representation space, but G-equivariant maps known as *intertwining operators.*

Definition 6 (Intertwining Operators). *Given two representations (π_1, V_1) and (π_2, V_2) of G, the space of intertwining operators is the space $Hom_G(V_1, V_2)$ of linear maps $\phi : V_1 \to V_2$ satisfying*

$$\phi \circ \pi_1(g) = \pi_2(g) \circ \phi$$

A basic fact about intertwining operators is

Theorem 2 (Schur's Lemma). *Given two finite-dimensional complex irreducible representations V_1, V_2 of a Lie group G, the intertwiners satisfy*

- $Hom_G(V_1, V_2) = \{0\}$ *(the zero map) if V_1 is not isomorphic to V_2.*
- $Hom_G(V_1, V_2) = \mathbf{C}$ *if V_1 is isomorphic to V_2.*

and one has the same fact for Lie algebra representations. Over a general field k, one gets that $Hom_G(V_1, V_1)$ (V_1 irreducible) can be a division algebra over k, not just k. In the case of infinite dimensional complex representations, if these are unitary Schur's lemma remains true, with the proof using the spectral theorem for self-adjoint operators. For Lie algebras, Schur's lemma remains true for arbitrary irreducible $U(\mathfrak{g})$ modules, even when infinite-dimensional[1].

Note that equivalence classes of representations form a ring

Definition 7 (Representation Ring)**.** *The representation ring $R(G)$ is the ring of equivalence classes $[V]$ of representations of G, with sum and product given by*

$$[V_1] + [V_2] = [V_1 \oplus V_2],\ [V_1] \cdot [V_2] = [V_1 \otimes V_2]$$

For Lie groups, one way to distinguish non-isomorphic representations is by their character:

Definition 8 (Character of a representation)**.** *The character χ_V of a representation (π, V) of G is the function on G*

$$\chi_V = Tr(\pi(g))$$

It gives a ring homomorphism from $R(G)$ to the ring of conjugation-invariant functions on G.

For infinite-dimensional representations, one can often make sense of the character as a distribution, defining the Harish-Chandra character on a function f by

$$\Theta_\pi(f) = Tr(\int_G f(g)\pi(g)dg)$$

For a Lie algebra $\mathfrak{g}$, one can define a different sort of invariant of a representation:

Definition 9. *If the center $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ acts by scalars on a representation (π, V), the representation is said to have infinitesimal character, and this is given by a homomorphism*

$$\chi_\pi : Z(\mathfrak{g}) \to C$$

with $\chi_\pi(z)$ the scalar by which z acts.

The infinitesimal character can be used to study Lie algebra representations that are not representations of the corresponding Lie group. Examples are the Verma modules we will begin studying next time. One source of Lie algebra representations that are not Lie group representations is functions on an open subset of a Lie group G that is invariant under $\mathfrak{g}$, but not under G. For representations that have both a distributional character and an infinitesimal character, the two are related by

$$\Theta_\pi(zf) = \chi_\pi(z)\Theta_\pi(f)$$

i.e. the characters is an eigendistribution for $Z(\mathfrak{g})$, with eigenvalues given by the infinitesimal character.

2 Lie algebra cohomology

An invariant of representations defined last semester is the Lie algebra cohomology $H^*(\mathfrak{g}, V)$. An abstract definition can be given as the derived functor of the invariants functor. A more concrete motivation comes from considering de Rham cohomology of a group, and taking coefficients in a representation. Important general properties of this are:

- Zero-dimensional cohomology of a representation is the invariant part of the representation:
$$H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$$

- $$H^1(\mathfrak{g}, \mathbf{C}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$$
This is the dual of the abelianization of $\mathfrak{g}$, zero for semi-simple $\mathfrak{g}$.

- Two dimensional cohomology $H^2(\mathfrak{g}, \mathbf{C})$ classifies central extensions
$$0 \to \mathbf{C} \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$
This is zero for semi-simple $\mathfrak{g}$, but important for affine Lie algebras, which are non-trivial central extensions of loop algebras.

- One dimensional cohomology classifies extensions of $\mathfrak{g}$ representations, i.e isomorphism classes of extensions
$$0 \to V_1 \to V \to V_2 \to 0$$
are given by
$$H^1(\mathfrak{g}, Hom_{\mathbf{C}}(V_2, V_1)) = H^1(\mathfrak{g}, V_2 \otimes_{\mathbf{C}} V_1) = Ext^1(V_2, V_1)$$

- For semi-simple Lie algebras, the invariants functor is exact: it takes exact sequences to exact sequences. One can prove this using the Casimir operator. One consequence of this is that for a semi-simple Lie algebra
$$H^i(\mathfrak{g}, V) = H^i(\mathfrak{g}, V^{\mathfrak{g}}) = H^i(\mathfrak{g}, \mathbf{C}) \otimes V^{\mathfrak{g}}$$
By the relation to de Rham cohomology, the $H^i(\mathfrak{g}, \mathbf{C})$ are just the topological cohomology groups of the compact Lie group G whose complexified Lie algebra is $\mathfrak{g}$. This is zero for $i = 1, 2$, non-zero for $i = 3$.

- A second important consequence is that for semi-simple Lie algebras, finite-dimensional representations are completely reducible since there are no non-trivial extensions. If $V_1 \subset V$ is a subrepresentation, so is V/V_1. Any finite dimensional representation V can be written as a direct sum of irreducibles V_i:
$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$
With Schur's lemma, the category of representations becomes quite simple, since there are no morphisms between irreducibles.

- We will be studying representations by looking at their highest weight spaces $V^{\mathfrak{n}^+} = H^0(\mathfrak{n}^+, V)$. The Lie algebra $\mathfrak{n}^+$ is not semi-simple and its invariants functor is not exact. The higher cohomology groups $H^*(\mathfrak{n}^+, V)$ will provide invariants of the representation.

References

[1] Knapp, A. *Lie Groups Beyond an Introduction, Second Edition* Birkhauser, 2002. Page 290.