Background on classification of Lie groups and Lie algebras

Math G4344, Spring 2012

This is the second half of a full year course on Lie groups and their representations. To first approximation I'll assume that Andrei Okounkov covered last semester everything you need to know about Lie groups and Lie algebras, so that this semester I can just discuss representation theory. I'll start though with a summary of basic facts about Lie groups and Lie algebras that Andrei should have covered, as well as some additional material to locate these facts in a more general mathematical context.

1 Lie groups

Definition 1 (Lie Group)**.** *A Lie group is a group object in the category of smooth (C^∞) manifolds*

In other words, a Lie group is a smooth manifold, with product and inverse maps that satisfy the group axioms and are smooth. We'll restrict attention to finite dimensional Lie groups, but infinite-dimensional Lie groups are quite interesting. A simple infinite dimensional example is a "loop group" LG, a group of maps $S^1 \to G$ from the circle to a finite dimensional Lie group, with point-wise multiplication.

It turns out (solution to Hilbert's 5th problem) that weakening the smoothness assumption leads to nothing new: starting with a topological manifold with product and inverse maps that are just continuous, one can show that continuity implies smoothness.

A much more concrete approach would be to define a Lie group as a "matrix group", a closed subgroup of $GL(n, \mathbf{C})$. This works fine for almost all the examples we will be considering, but there is an exception. The group $Sp(2n, \mathbf{R})$ of linear transformations preserving a symplectic form has a non-trivial double-cover that we will be studying, called the "metaplectic group". This group has no finite-dimensional faithful representations, the interesting representations will be infinite dimensional, so it cannot be thought of as a group of finite-dimensional matrices.

Alternatively, one can do algebraic geometry instead of differential geometry, and study "algebraic groups", group objects in the category of schemes (or some other theory of algebraic varieties). This allows one to discuss groups like $GL(n, k)$ for different fields (e.g. $k = \mathbf{Q}_p$, the p-adic numbers). Except for things like the metaplectic group example, we'll be working with groups that actually are algebraic groups over $\mathbf{R}$ or its algebraic closure $\mathbf{C}$, but will not try and think of these as algebraic groups.

2 Lie algebras

Given a Lie group G, one can form an algebra that reflects its infinitesimal structure, the algebra $\mathfrak{g}$ of left-invariant vector fields. This will be a vector space carrying a bracket operation, the Lie bracket of two vector fields. This motivates a more general definition of a Lie algebra:

Definition 2 (Lie Algebra)**.** *A Lie algebra $\mathfrak{g}$ is a vector space over a field k, with a binary operation $[\cdot,\cdot]$ that is bilinear, antisymmetric, and satisfies the Jacobi identity*

$$[[X,Y]Z]+[[Z,X],Y]+[[Y,Z],X]=0$$

for all elements $X,Y,Z\in\mathfrak{g}$

This is a non-associative algebra. Another way of packaging the same structure is as an associative algebra, the universal enveloping algebra $U(\mathfrak{g})$. One way to contruct this is to take the tensor algebra of $\mathfrak{g}$ and impose a relation reflecting the bracket:

Definition 3. *The universal enveloping algebra $U(\mathfrak{g})$ is the associative algebra*

$$T^*(\mathfrak{g})/I$$

where $T^*(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and I is the two-sided ideal generated by the relations

$$X\otimes Y-Y\otimes X=[X,Y]$$

for elements $X,Y\in\mathfrak{g}\subset T^*(\mathfrak{g})$.

We will only be using the cases $k=\mathbf{R}$ or $\mathbf{C}$. Our Lie algebras will be finite dimensional. An interesting class of infinite-dimensional Lie algebras are the affine Kac-Moody Lie algebras. These are the Lie algebras corresponding to central extensions of loop groups.

The Poincaré-Birkhoff-Witt theorem says that ordered powers of a basis of $\mathfrak{g}$ give a basis of $U(\mathfrak{g})$, and this implies that the inclusion of $\mathfrak{g}$ in $U(\mathfrak{g})$ is injective. The algebra $U(g)$ is a filtered algebra, with filtration inherited from the grading on the tensor algebra

$$k\subset\mathfrak{g}=U^1(\mathfrak{g})\subset U^2(\mathfrak{g})\subset\cdots$$

and the associated graded algebra

$$GrU(\mathfrak{g})=\bigoplus_i U^i(\mathfrak{g})/U^{i-1}(\mathfrak{g})$$

is the symmetric algebra $S^*(\mathfrak{g})$. One often will try to prove things about $U(\mathfrak{g})$ by doing so for the commutative algebra $S^*(\mathfrak{g})$ and then lifting to $U^*(\mathfrak{g})$.

The classification of Lie algebras and Lie groups is closely related, since:

Theorem 1 (Fundamental theorem of Lie theory)**.** *The category of finite dimensional Lie algebras is equivalent to the category of connected, simply-connected Lie groups.*

To a Lie group is associated a single Lie algebra, but several Lie groups may have the same Lie algebra. One of these will be the simply connected one.

Examples: $Spin(n, \mathbf{C})$ is simply connected double cover of $SO(n, \mathbf{C})$, $SL(n, \mathbf{C})$ is the simply-connected n-fold cover of $PSL(n, \mathbf{C})$.

If $\mathfrak{g}$ is the Lie algebra of the Lie group G, then $U(\mathfrak{g})$ will be the algebra of left-invariant differential operators on G. An alternative characterization is as the convolution algebra of distributions supported at the identity of G.

Even when a Lie group G has trivial center, $U(\mathfrak{g})$ will generally have an interesting center $Z(\mathfrak{g})$. When there is an invariant bilinear form on $\mathfrak{g}$, we get an associated quadratic element in $Z(\mathfrak{g})$, the Casimir element.

3 Classification of Lie algebras

There are two important classes of Lie algebras, with very different behavior: solvable Lie algebras and semi-simple Lie algebras.

Definition 4 (Solvable Lie algebra). *A Lie algebra $\mathfrak{g}$ is solvable if there exists a sequence of Lie subalgebras*

$$0 \subset \cdots \subset \mathfrak{g}_2 \subset \mathfrak{g}_1 \subset \mathfrak{g}_0 = \mathfrak{g}$$

such that for all i, $\mathfrak{g}_{i+1}$ is an ideal in $\mathfrak{g}_i$ and the quotient Lie algebra $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ is abelian.

The standard example to keep in mind is the Lie algebra of upper triangular matrices. Other examples that we will see include the Heisenberg Lie algebra and Borel subalgebras.

Definition 5 (Semi-simple Lie algebra). *A Lie algebra $\mathfrak{g}$ is semi-simple if it contains no non-zero solvable ideals*

Examples: $\mathfrak{sl}(n)$, the Lie algebra of trace-free matrices, as well as the Lie algebras of compact simple Lie groups ($\mathfrak{su}(n), \mathfrak{so}(n)$, etc.).

Theorem 2 (Levi Decomposition). *A Lie algebra can be decomposed as a direct sum*

$$\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{g}'$$

where $rad(\mathfrak{g})$ is a (unique) maximal solvable ideal (the "radical" of $\mathfrak{g}$) and $\mathfrak{g}'$ is semi-simple.

A more general class that one often wants to consider are Lie algebras where $\text{rad}(\mathfrak{g})$ is rather trivial, just the center of $\mathfrak{g}$.

Definition 6 (Reductive Lie algebra). *A Lie algebra is reductive if $rad(\mathfrak{g}) = z(\mathfrak{g})$. Any reductive Lie algebra is a sum*

$$\mathfrak{g} = z(\mathfrak{g}) \oplus \mathfrak{g}'$$

where $z(\mathfrak{g})$ is an Abelian subalgebra (the center of $\mathfrak{g}$, note this is NOT the same as $Z(\mathfrak{g}) \subset U(\mathfrak{g})$) and $\mathfrak{g}'$ is semi-simple.

Examples: Reductive Lie algebras include $\mathfrak{gl}(n)$ and $\mathfrak{u}(n)$.

The classification of complex semi-simple Lie algebras $\mathfrak{g}$ relies upon the existence in this case of a Cartan subalgebra $\mathfrak{h}$, which is a maximal toral (commutative with semi-simple elements) subalgebra. Its dimension is the rank of the algebra (and of the corresponding Lie group). All possible choices of $\mathfrak{h}$ are related by conjugation. Note that this fails over other fields, e.g. $\mathbf{R}$.

The adjoint action of $\mathfrak{h}$ is used to decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where the root space $\mathfrak{g}_\alpha$ is the eigenspace of the $\mathfrak{h}$ action with eigenvalue $\alpha \in \mathfrak{h}^*$, and Δ is the set of roots α.

For any Lie algebra one can define an invariant bilinear form, the Killing form:

Definition 7. *The Killing form of a Lie algebra $\mathfrak{g}$ is defined by*

$$< X, Y >= tr(ad(X)ad(Y))$$

where ad is the adjoint representation.

For a complex simple Lie algebra this provides a non-degenerate bilinear form. Restricting to the Cartan subalgebra $\mathfrak{h}$, this remains non-degenerate, so can be used to define an isomorphism $\mathfrak{h} = \mathfrak{h}^*$ and thus a non-degenerate bilinear form on $\mathfrak{h}^*$. This is positive definite on non-zero elements of $\mathfrak{h}_{\mathbf{R}}$, the subspace of real linear combinations of the the roots in Δ.

For each $\alpha \in \Delta$, one gets a reflection map s_α on $\mathfrak{h}^*$ that takes $\alpha \to -\alpha$, preserving the Killing form. These generate a finite group of permutations of the roots, the Weyl group $W(\mathfrak{g}, \mathfrak{h})$.

The set Δ of roots can be decomposed as $\Delta = \Delta^+ + \Delta^-$, such that

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \text{ and } \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$$

are (nilpotent) Lie subalgebras, with $\alpha \in \Delta^-$ if $-\alpha \in \Delta^+$. Taking

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$$

gives an important subalgebra of $\mathfrak{g}$ called a Borel subalgebra (for which $\mathfrak{n}^+$ is the "nilpotent radical"). $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$.

The space of Borel subalgebras of $\mathfrak{g}$ can be identified with G/B, where G, B are the Lie groups corresponding to $\mathfrak{g}, \mathfrak{b}$. This is a complex projective variety and will play a crucial role in the study of the representation theory of G via geometry. It is often known as the "flag variety", since it parametrizes flags in $\mathbf{C}^n$ in the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$.

A generalization is the notion of a "parabolic subalgebra". This is a Lie algebra $\mathfrak{p}$ such that $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$, with corresponding Lie group P. G/P is also a complex projective variety.

One ends up with the following list that classifies complex simple Lie algebras (the subscript n gives the rank):

- $A_n, n = 1, 2, 3, \cdots$, $\mathfrak{sl}(n+1, \mathbf{C})$
- $B_n, n = 2, 3, 4, \cdots$, $\mathfrak{so}(2n+1, \mathbf{C})$
- $C_n, n = 2, 3, 4, \cdots$, $\mathfrak{sp}(2n, \mathbf{C})$
- $D_n, n = 4, 5, 6, \cdots$, $\mathfrak{so}(2n, \mathbf{C})$
- G_2, F_4, E_6, E_7, E_8, corresponding exceptional Lie algebras

In this course we'll continually use the A_n example and consider its representations in detail, so you should become familiar with how things work in that case. We'll also cover in some detail later the B_n and D_n case, but from the perspective of the spin representation. When we discuss the highest weight theory of finite dimensional representations, we'll review the story of the Weyl group and how it acts on $\mathfrak{h}^*$.

4 Lie algebras over other fields

The classification of semi-simple Lie algebras over $\mathbf{R}$ is quite a bit more complicated. For each complex semi-simple Lie algebra $\mathfrak{g}$ there will be multiple non-isomorphic "real forms", these are real Lie algebras $\mathfrak{g}_\mathbf{R}$ such that

$$\mathfrak{g} = \mathfrak{g}_\mathbf{R} \otimes \mathbf{C} = \mathfrak{g}_\mathbf{R} \oplus i\mathfrak{g}_\mathbf{R}$$

Examples: Real forms of $\mathfrak{sl}(2, \mathbf{C})$ are $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbf{R})$. Real forms of $\mathfrak{so}(n, \mathbf{C})$ include Lie algebras of orthogonal groups for quadratic forms of different signatures: $\mathfrak{so}(p, q, \mathbf{R})$ for $(p + q = n)$.

It turns out that there will always be one "compact" real form, which corresponds to a compact Lie group. We will always be using this specific real form until later in the course, when we will deal with just one example of a different real form, with a non-compact Lie group, $SL(2, \mathbf{R})$.

Remarkably, it turns out that one can find not just real forms for a complex semi-simple Lie algebra, but a $\mathbf{Z}$-form, using a basis for the Lie algebra due to Chevalley, in which all the defining relations of the Lie algebra have $\mathbf{Z}$ coefficients. The group of adjoint transformations of the Lie algebra is then an algebraic group defined over $\mathbf{Z}$. This means that one can use it to define a group over any commutative ring, giving for each complex Lie algebra a wide range of different kinds of groups to study. For example, for $\mathfrak{gl}(2, \mathbf{C})$ the adjoint group is $PGL(2, \mathbf{C})$ and one can construct and study groups like $PGL(2, \mathbf{F}_q)$, where $\mathbf{F}_q$ is a finite field.